Structural Set Theory in Foundations

The Elementary Theory of the Category of Sets (ETCS)

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Introduction

$$\text{“is } 3 \in \mathbb{N}\text{?”}$$

“is $3 \in \mathbb{N}$?” ✓

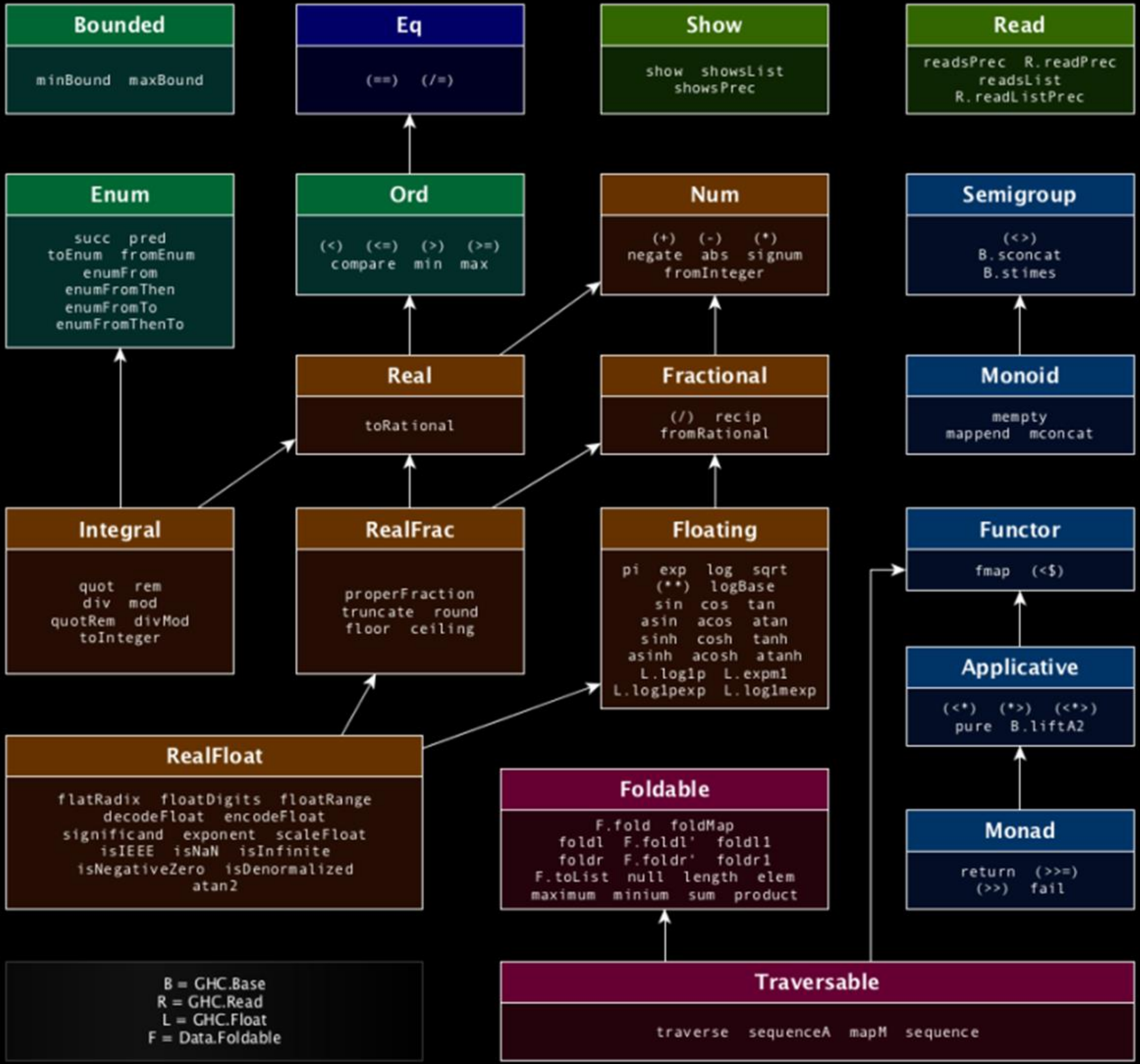
“is $0 \in \mathbb{N}$?”

“is $0 \in \mathbb{N}$?” ✓

"is $\sqrt{2} \in \mathbb{Q}$?"

“is $\sqrt{2} \in \mathbb{Q}$?” ✗

“is $\pi \in \log$?”


Bounded
minBound maxBound
Eq
(==) (/=)
Show
show showsList
showsPrec
Read
readsPrec R.readPrec
readsList
R.readListPrec
Enum
succ pred
toEnum fromEnum
enumFrom
enumFromThen
enumFromTo
enumFromThenTo
Ord
(<) (<=) (>) (>=)
compare min max
Num
(+) (-) (*)
negate abs signum
fromInteger
Semigroup
(<>)
B.sconcat
B.stimes
Real
toRational
Fractional
(/) recip
fromRational
Monoid
mempty
mappend mconcat
Integral
quot rem
div mod
quotRem divMod
toInteger
RealFrac
properFraction
truncate round
floor ceiling
Floating
pi exp log sqrt
(**) logBase
sin cos tan
asin acos atan
sinh cosh tanh
asinh acosh atanh
L.log1p L.expm1
L.log1pexp L.log1mexp
Functor
fmap (<$)
Applicative
(<*) (*>) (<*>)
pure B.liftA2
RealFloat
flatRadix floatDigits floatRange
decodeFloat encodeFloat
significand exponent scaleFloat
isIEEE isNaN isInfinite
isNegativeZero isDenormalized
atan2
Foldable
F.fold foldMap
foldl F.foldl' foldl1
foldr F.foldr' foldr1
F.toList null length elem
maximum minium sum product
Monad
return (>>=)
(>>) fail
B = GHC.Base
R = GHC.Read
L = GHC.Float
F = Data.Foldable
Traversable
traverse sequenceA mapM sequence

- Is [0,1] closed?
- Is $\mathbb{Z}$ a group?
- What is the fundamental group of $\mathbb{R}\setminus\{0\}$?
- What is the Fourier series of $\sin(x) + \sin(\pi x)$?

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- What is the Fourier series of $\sin(x) + \sin(\pi x)$?
- Is a rectangle prime?
- Is 3 surjective?
- Does a prime converge?

$$A \cdot B$$

$$A \cdot B$$

$A, B : \texttt{Int}$ $\quad A \cdot B := A \cdot_{\mathbb{Z}} B$

$A, B : \texttt{Matrix}$ $\quad A \cdot B := \sum a_{ij} b_{jk}$

$A, B : \texttt{Tuple}$ $\quad A \cdot B := [a_1, a_2, \ldots, b_1, b_2, \ldots]$

$A, B : \texttt{Path}$ $\quad A \cdot B := t \mapsto \begin{cases} A(2t) & t \in [0, 1/2] \\ B(2t-1) & t \in [1/2, 1] \end{cases}$

$\vdots \qquad \vdots$

In ZFC:

Everything is a set

In ZFC:

$\in$ is a global relation on sets, so

$$A \in B$$

is a well-formed proposition
for all sets A, B.

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ZFC is *single-sorted*

Axiom of Regularity (ZFC):

Every non-empty set X has an element x disjoint from itself:

$$\forall X \left(X \neq \emptyset \to \exists x (x \in X \wedge x \cap X = \emptyset) \right)$$

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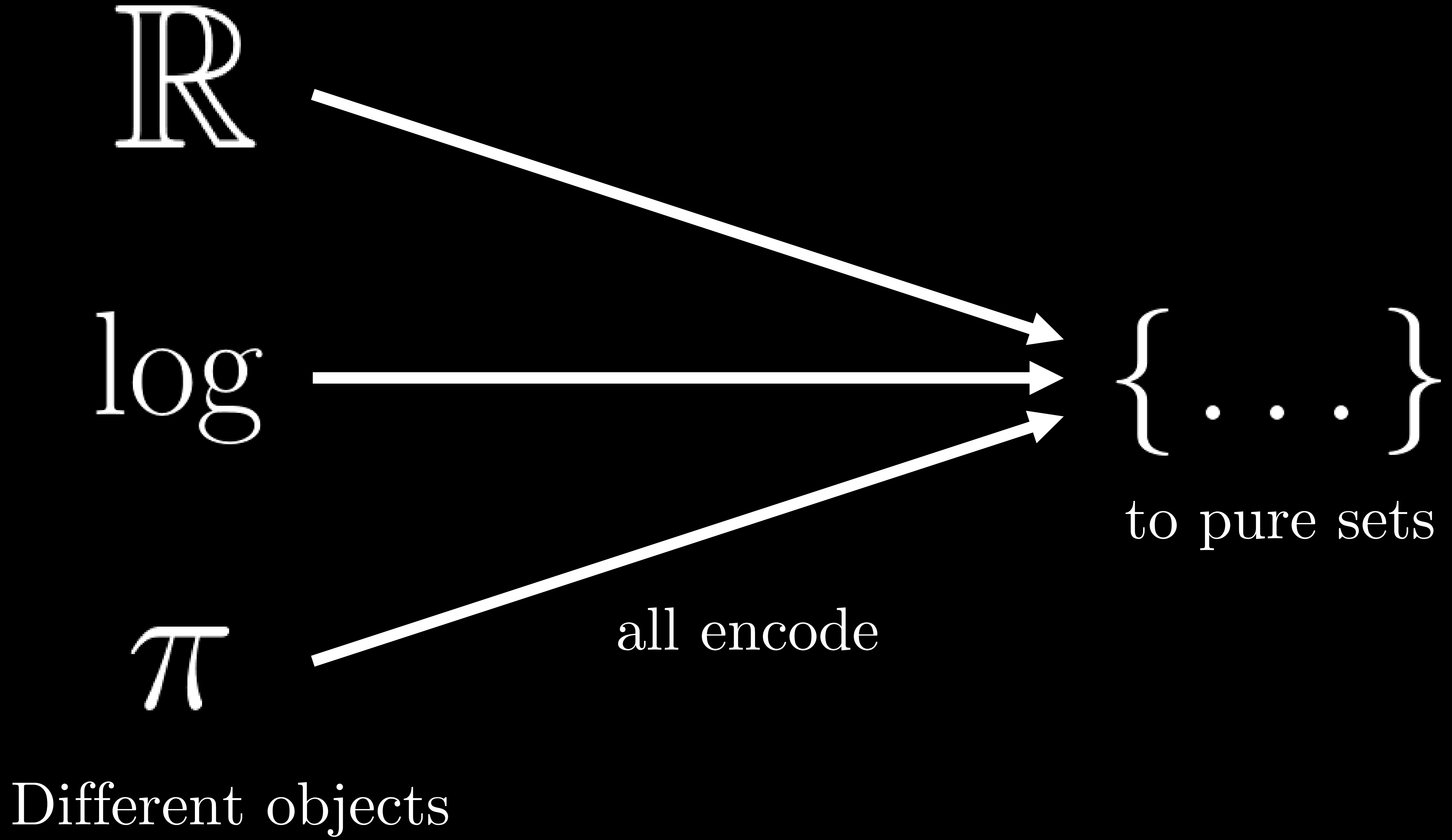
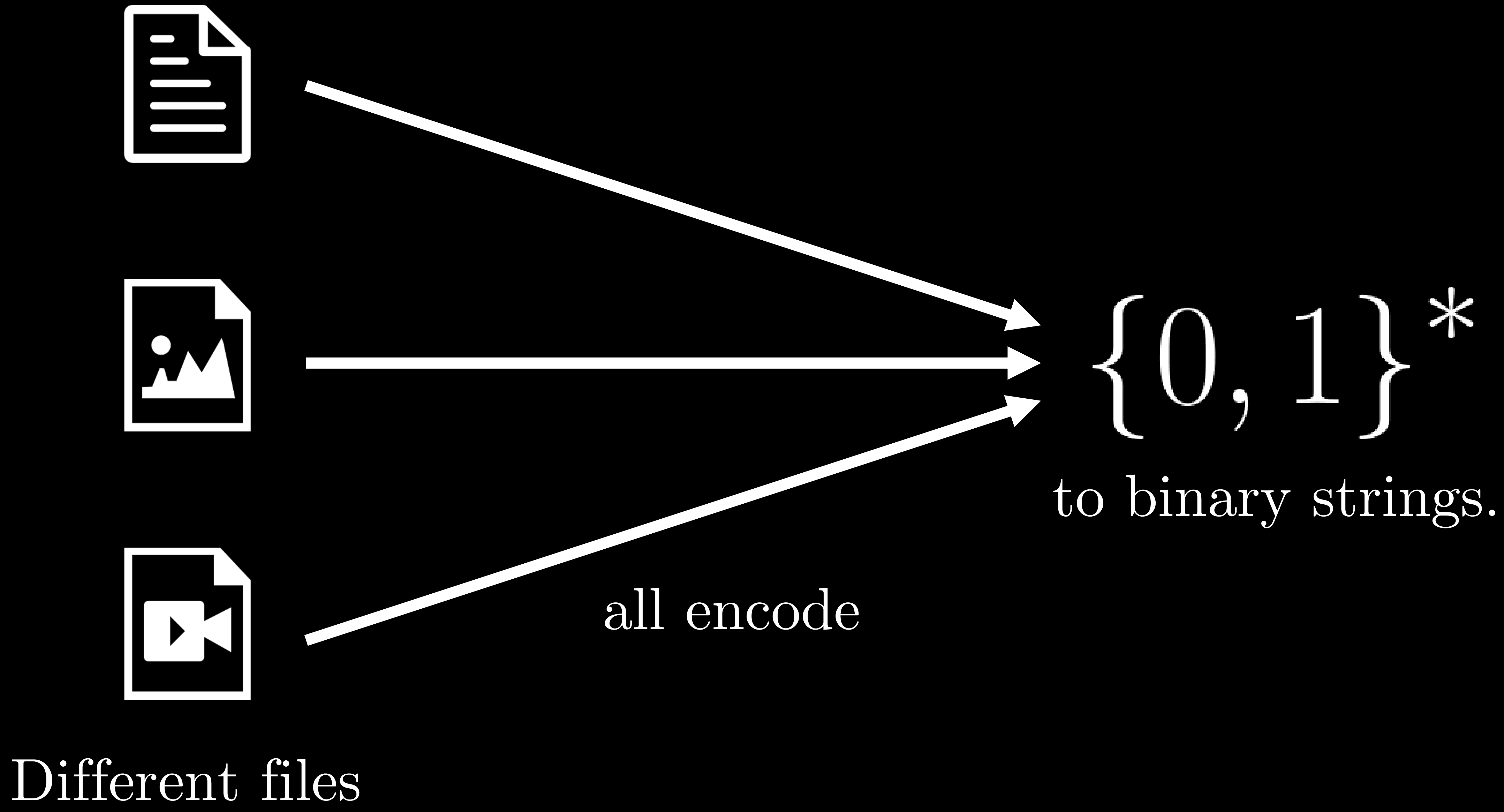
Is $3 \cap \mathbb{R}$ empty?

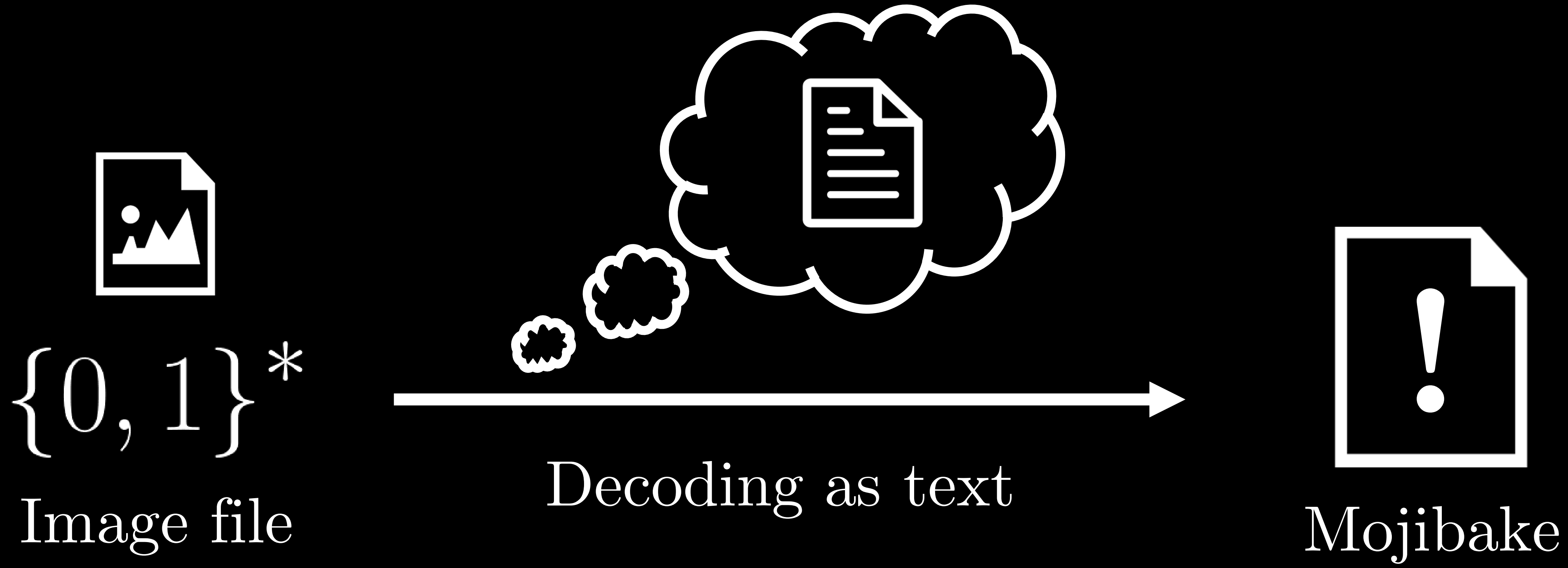
What does an element of this even look like?

ZFC also includes a set of standard encodings of mathematical objects

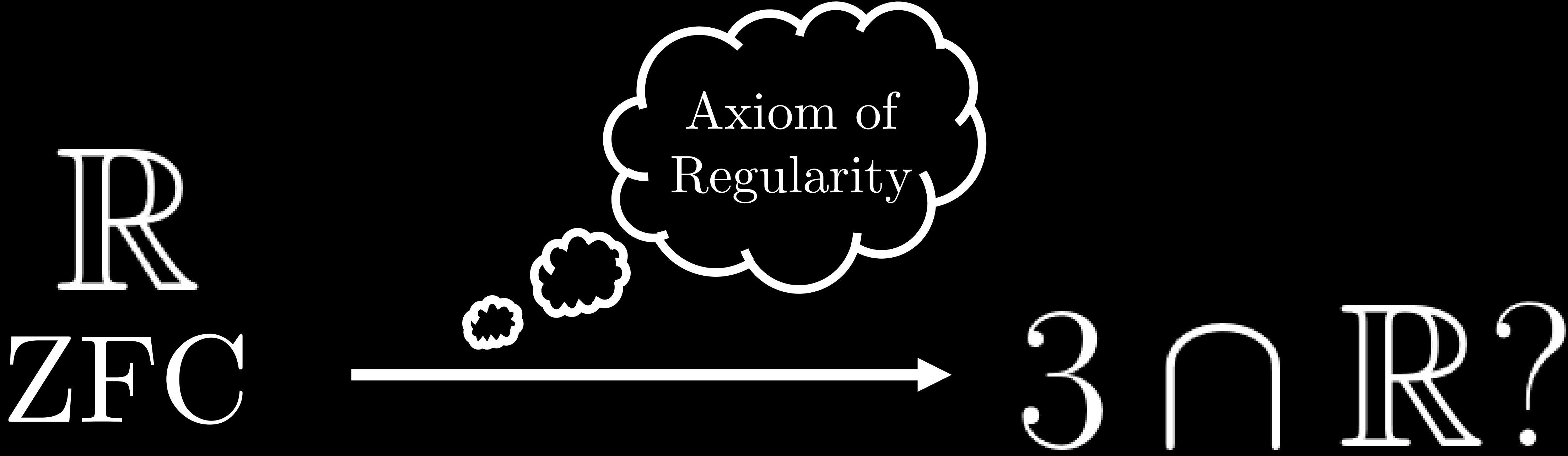

$\mathbb{R}$
$\log$
π
Different objects
all encode
$\{\ldots\}$
to pure sets



$\{0,1\}^*$
to binary strings.
all encode
Different files


{0, 1}*
Image file
Decoding as text
Mojibake

Using the wrong encodings result in meaningless outputs, but that doesn't mean files/encodings are useless.



Some of the axioms/encodings result in meaningless questions, but that doesn't mean sets/encodings are useless.

“is $3 \in 17$?”

$$N$$

The (set of) *natural numbers*

$$0$$

The natural number *zero*

$$s : N \to N$$

The *successor function*

$$s(n) := n \cup \{n\}$$

The *successor function*

The "vulgar" way	Axiomatic foundations
counting	cardinality
addition	simple recursion
less-than	well-ordering

$$N, \quad 0, \quad s$$

“is $3 \in 17$?”

$$s_{\text{Johnny}}(n) := n \cup \{n\}$$

$$17 = \{0, 1, 2, \ldots, 16\}$$

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$$17 = \{0, 1, 2, \ldots, 16\}$$

$$\text{so } \ 3 \in 17$$

$$s_{\text{Ernie}}(n) := \{n\}$$

$$17 = \{16\}$$

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$$17 = \{16\}$$

$$\text{so } 3 \notin 17$$

Proposition. *A set X has n elements if and only if there is a bijection between X and the natural number n.*

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$$s_{\text{Johnny}}(n) := n \cup \{n\} \qquad s_{\text{Ernie}}(n) := \{n\}$$

$$n = \{0, 1, 2, \ldots, n-1\} \qquad n = \{n-1\}$$

Proposition. *A set X has n elements if and only if there is a bijection between X and the natural number n.*

$$s_{\text{Johnny}}(n) := n \cup \{n\}$$
$$n = \{0, 1, 2, \ldots, n-1\}$$

$$s_{\text{Ernie}}(n) := \{n\}$$
$$n = \{n-1\}$$

True for Johnny...

Proposition. *A set X has n elements if and only if there is a bijection between X and the natural number n.*

$$s_{\text{Johnny}}(n) := n \cup \{n\}$$

$$n = \{0, 1, 2, \ldots, n-1\}$$

True for Johnny...

$$s_{\text{Ernie}}(n) := \{n\}$$

$$n = \{n-1\}$$

but not for Ernie.

$$s_{\text{Johnny}}(n) := n \cup \{n\}$$

$$s_{\text{Ernie}}(n) := \{n\}$$

$$\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} = 3 = \{\{\{\emptyset\}\}\}$$

$$s_{\text{Johnny}}(n) := n \cup \{n\} \qquad s_{\text{Ernie}}(n) := \{n\}$$

At least one of these must be “wrong”...

Set-theoretic Platonism:

There is a "true" account; there is a particular set that is the "real" set of natural numbers.

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That is, there is a "correct" assignment of sets to

$$N, \quad 0, \quad s$$

and all other assignments are wrong.

"...if the number 3 is really one set rather than another, it must be possible to give some cogent reason for thinking so; for the position that this is an unknowable truth is hardly tenable.

But there seems to be little to choose among the accounts. Relative to our purposes in giving an account of these matters, one will do as well as another, stylistic preferences aside.”

Structuralism

A *vector space* over a field, K, is a set, V, along with two maps, $+ : V^2 \to V$ and $\cdot : K \times V \to V$, called *vector addition* and *scalar multiplication*, respectively, that satisfies the following axioms for all $\mathbf{u},\mathbf{v},\mathbf{w} \in V$ and $a,b \in K$:

(V1) $(V,+)$ is an abelian group.

- (A1) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (commutativity of vector addition);
- (A2) $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ (associative of vector addition);
- (A3) $\exists \mathbf{0}_V$ such that $\mathbf{v} + \mathbf{0}_V = \mathbf{0}_V + \mathbf{v} = \mathbf{v}$ (existence of vector additive identity);
- (A4) $\exists (-\mathbf{v}) \in V$ such that $\mathbf{v} + (-\mathbf{v}) = (-\mathbf{v}) + \mathbf{v} = \mathbf{0}_V$ (existence of vector addition inverses);
- (A5) $\mathbf{u} + \mathbf{v} \in V$ (closure of vector addition).

(V2) $a \cdot (\mathbf{u} + \mathbf{v}) = a \cdot \mathbf{u} + b \cdot \mathbf{v}$ (distributivity of scalar multiplication over vector addition);

(V3) $(a + b) \cdot \mathbf{v} = a \cdot \mathbf{v} + b \cdot \mathbf{v}$ (distributivity of scalar multiplication over field addition);

(V4) $(ab) \cdot \mathbf{v} = a \cdot (b\mathbf{v})$ (compatibility of scalar multiplication with field multiplication);

(V5) $1_K \cdot \mathbf{v} = \mathbf{v}$ (existence of scalar multiplicative identity).

Structural Set Theory

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$$3 \stackrel{?}{=} \{\{\{\emptyset\}\}\}$$

Structural Set Theory

What is 3?

Structural Set Theory

What is 3? ✗

Structural Set Theory

What are *all* the natural numbers?

What is 3?

Structural Set Theory

What *structure* is the natural numbers?

What is 3?

Structural Set Theory

What *structure* is the natural numbers?

What is 3?

Primitive Notions

Material Set Theories

axiomatise

Sets

&

Membership

function

Input object $\rightsquigarrow$ One output

Material approach:

Represent a function $f : A \to B$ as the relation

$$\hat{f} = \{(a, b) : b \text{ is the } f\text{-image of } a\} \subseteq A \times B$$

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Conversely, a relation R satisfies the property that

$$\big((x, y) \in R \wedge (x, z) \in R\big) \to y = z$$

then R is the representation of some function.

Material approach:

A function *is* a relation R satisfies the property that

$$\big((x,y) \in R \wedge (x,z) \in R\big) \rightarrow y = z$$

$$\operatorname{dom}(f) := \left\{ x \,\middle|\, \exists y : (x, y) \in f \right\}$$

$$\operatorname{im}(f) := \left\{ y \,\middle|\, \exists x : (x, y) \in f \right\}$$

$$\operatorname{cdm}(f) := \;???$$

Let $A \subset B$ and consider the functions

$$\mathrm{id}_A : A \to A \qquad\qquad \iota_A : A \hookrightarrow B$$

both defined by $x \mapsto x$.

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$$\mathrm{id}_A : A \to A \qquad\qquad \iota_A : A \hookrightarrow B$$

both defined by $x \mapsto x$. Then,

$$\mathrm{id}_A = \{(x, x) : x \in A\} = \iota_A$$

Material Set Theories

axiomatise

Sets

&

Membership

Structural Set Theories

axiomatise

Sets

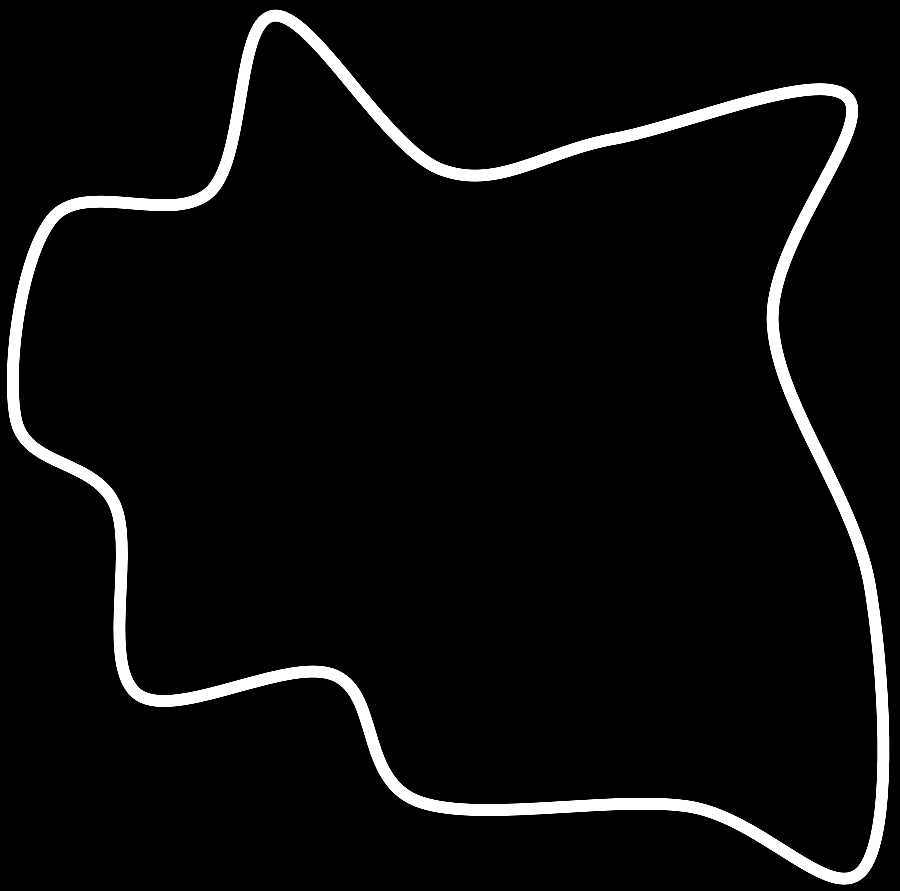
&

Functions

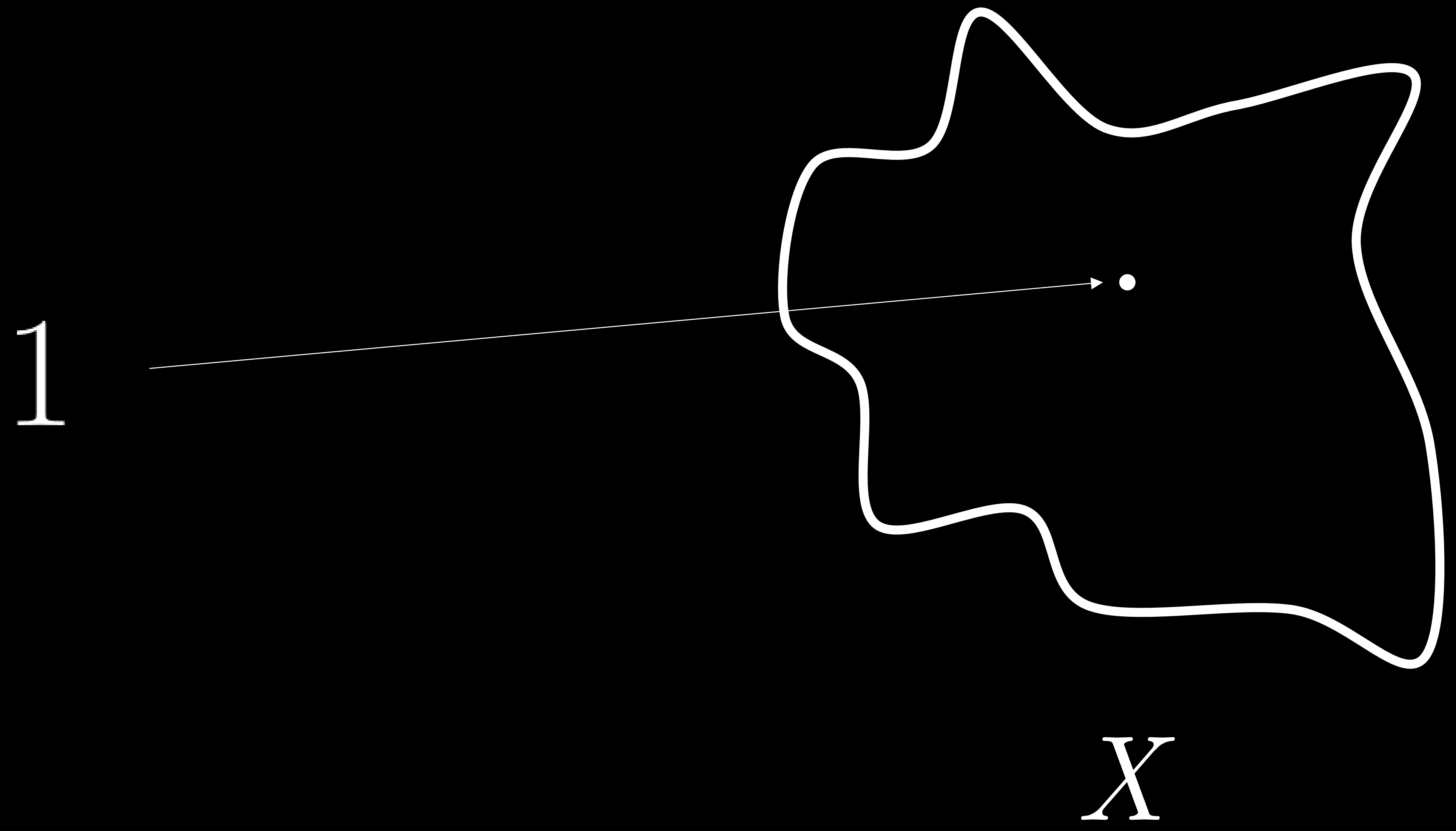
The Yoneda Lemma

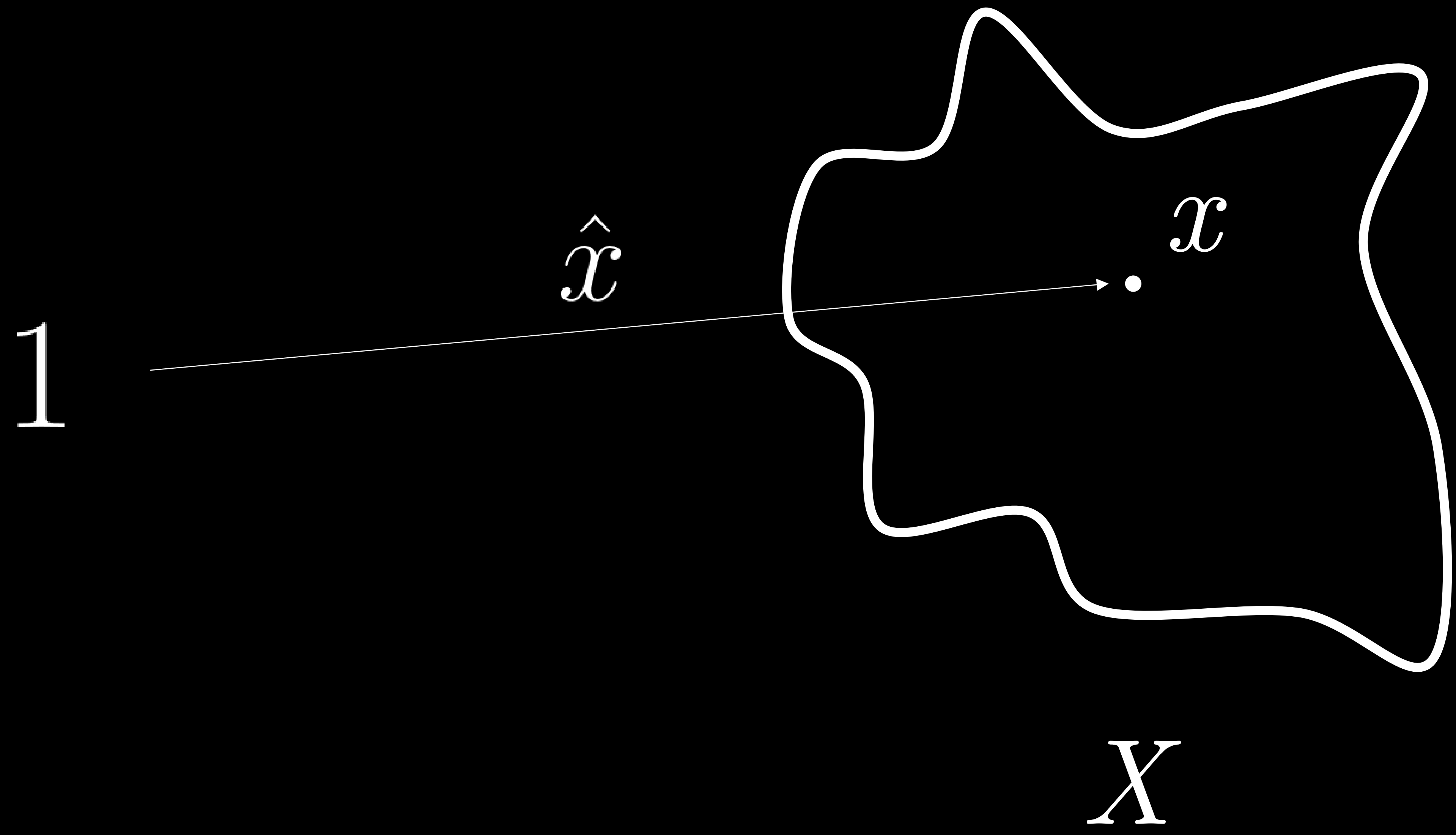
$$1 = \{\bullet\}$$

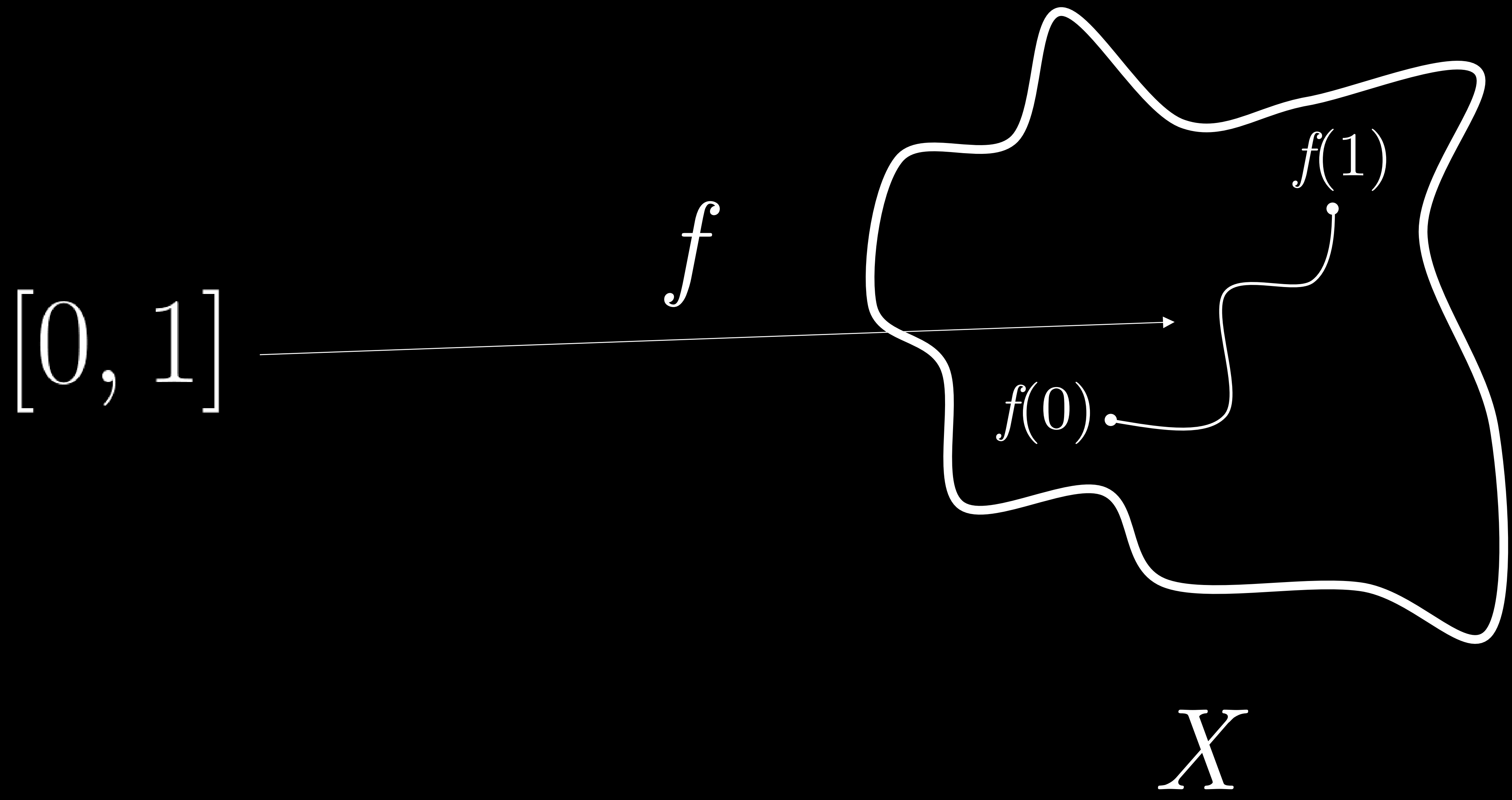
1



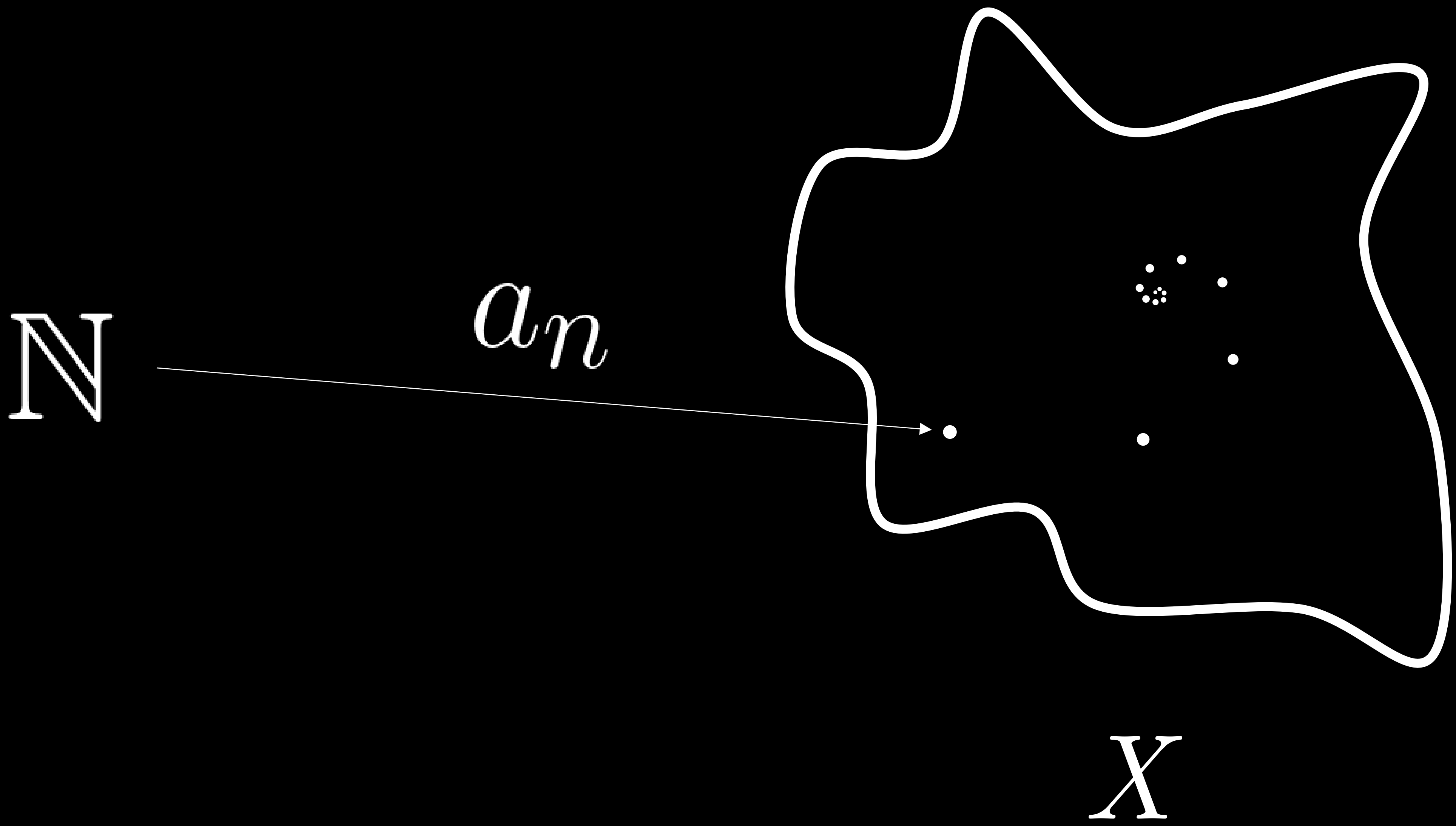
X

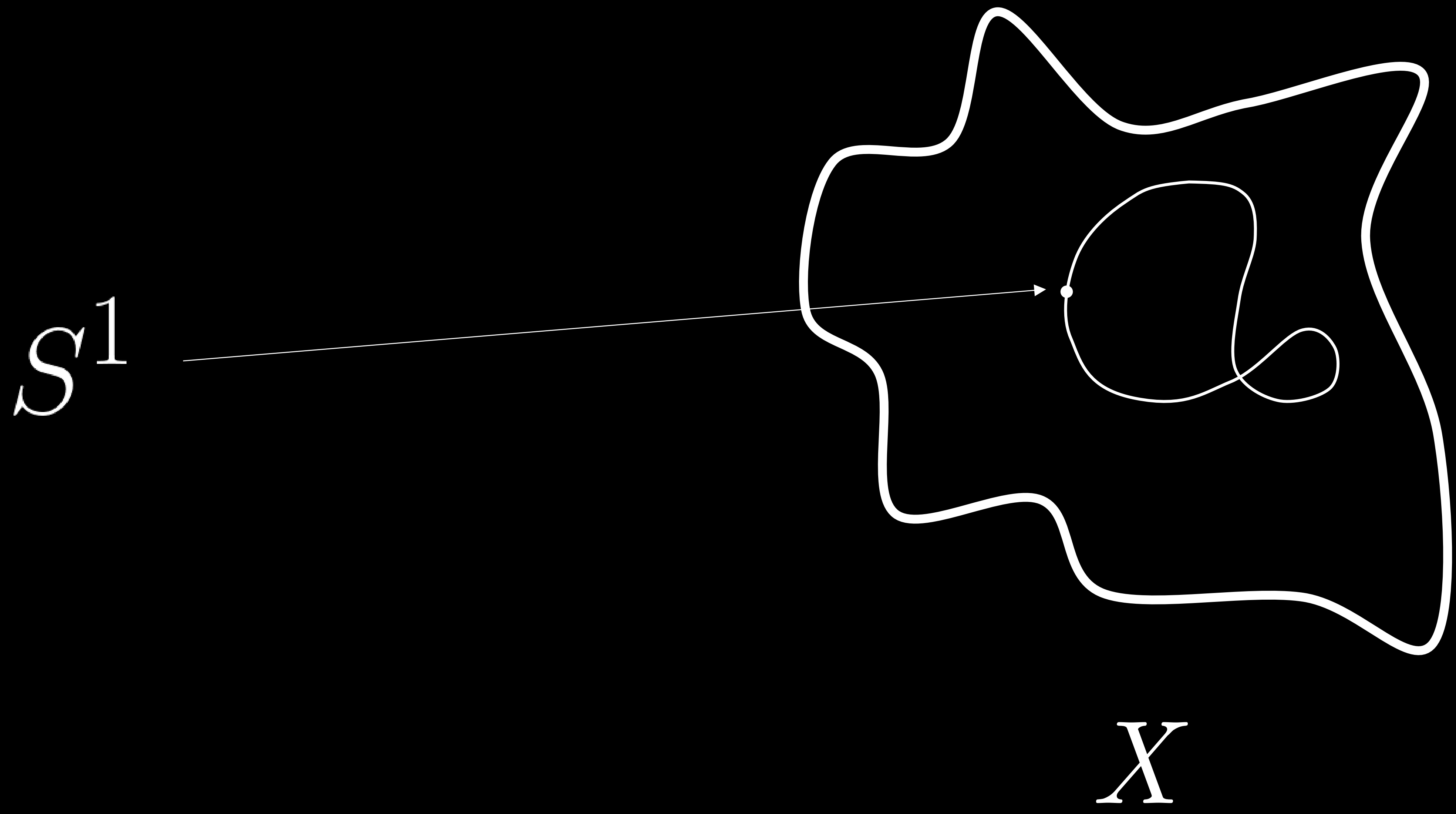

1
X


1
$\hat{x}$
x
X



$[0,1]$
f
$f(1)$
$f(0)$
X


$\mathbb{N}$
a_n
X


S^1
X

$$S \to X$$

$$\Updownarrow$$

"Generalised element of X of shape S"

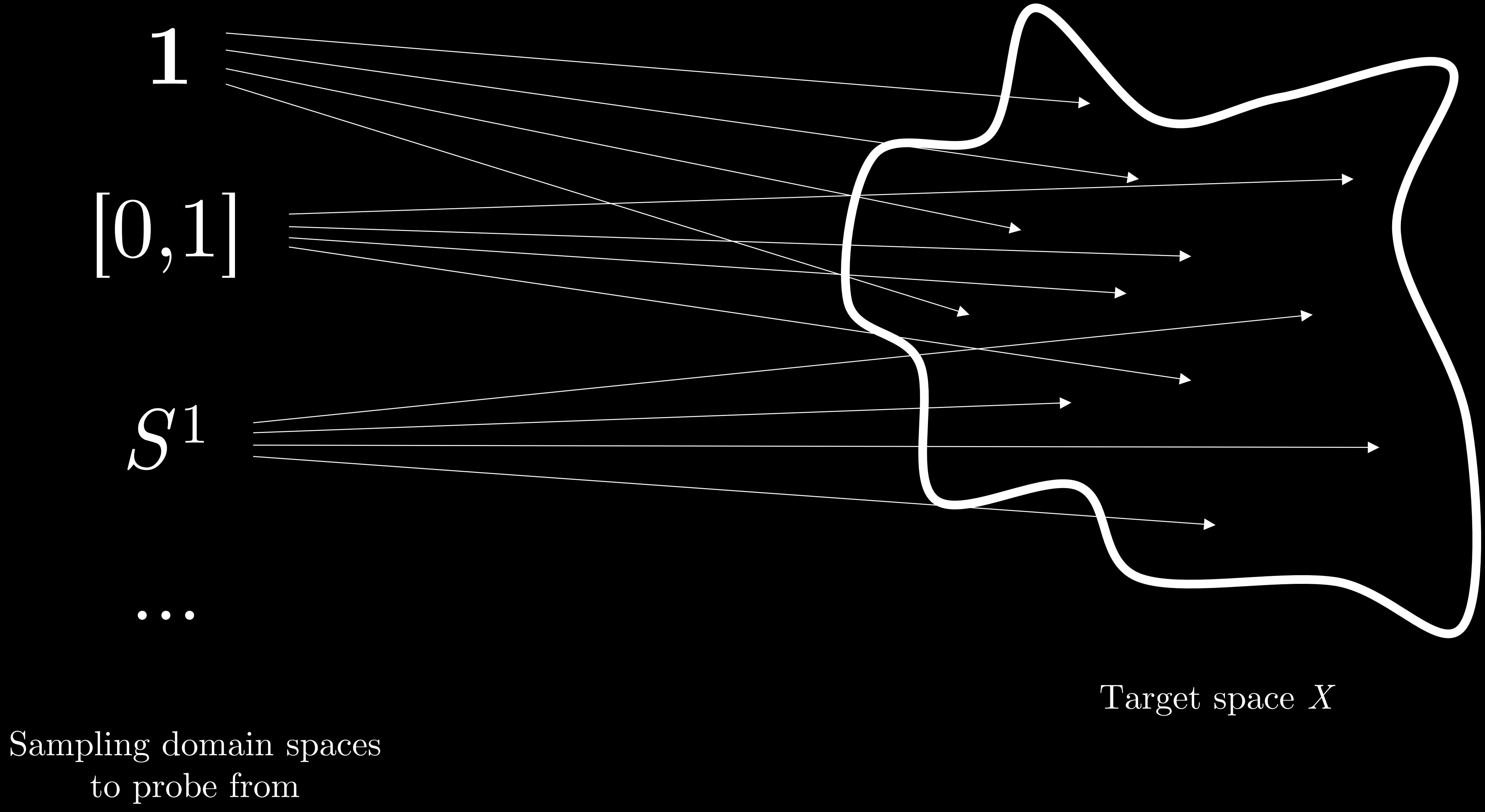
$$X = \{S, \mathcal{T}\}$$

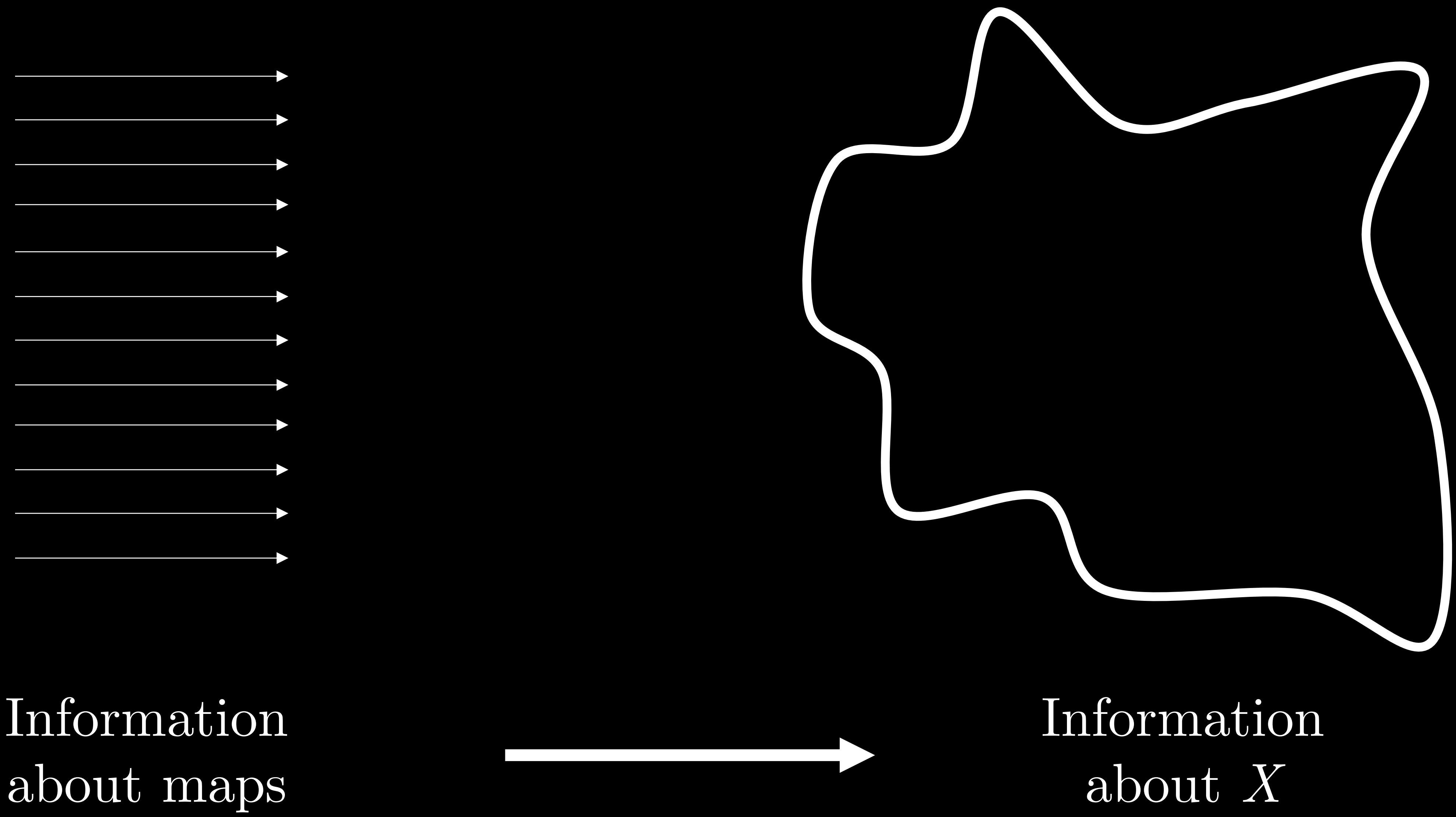
$$X = \{S, \mathcal{T}\}$$

$$\{1 \to X\} \cong S$$

$$\{[0,1] \to X\} \rightsquigarrow H_0(X)$$

$$\{S^1 \to X\} \rightsquigarrow \pi_1(X)$$


1
[0,1]
S^1
...
Target space X
Sampling domain spaces
to probe from


Information about maps
Information about X

$$\begin{matrix}\text{Information} \\ \text{about maps}\end{matrix} \cong \begin{matrix}\text{Information} \\ \text{about } X\end{matrix}$$

Lemma (Yoneda). *Let $\mathcal{C}$ be a locally small category. Then,*

$$\hom_{[\mathcal{C},\mathbf{Set}]}(H_A, F) \cong F(A)$$

naturally in $F \in \mathrm{ob}([\mathcal{C},\mathbf{Set}])$ and $A \in \mathrm{ob}(\mathcal{C})$.

Corollary.

$$X \cong Y \text{ \textit{if and only if} } \hom(X, -) \cong \hom(Y, -)$$

Subobjects

$$\{1\},\ \{2\},\ \{\text{cat}\}$$

$$X := \{1, 2, 3\}$$

$$\{1\} \cong \{2\} \cong \{\text{cat}\}$$

$$\{1\} \subseteq X$$

$$\{2\} \subseteq X$$

$$\{\text{cat}\} \not\subseteq X$$

$$A \rightarrowtail X$$

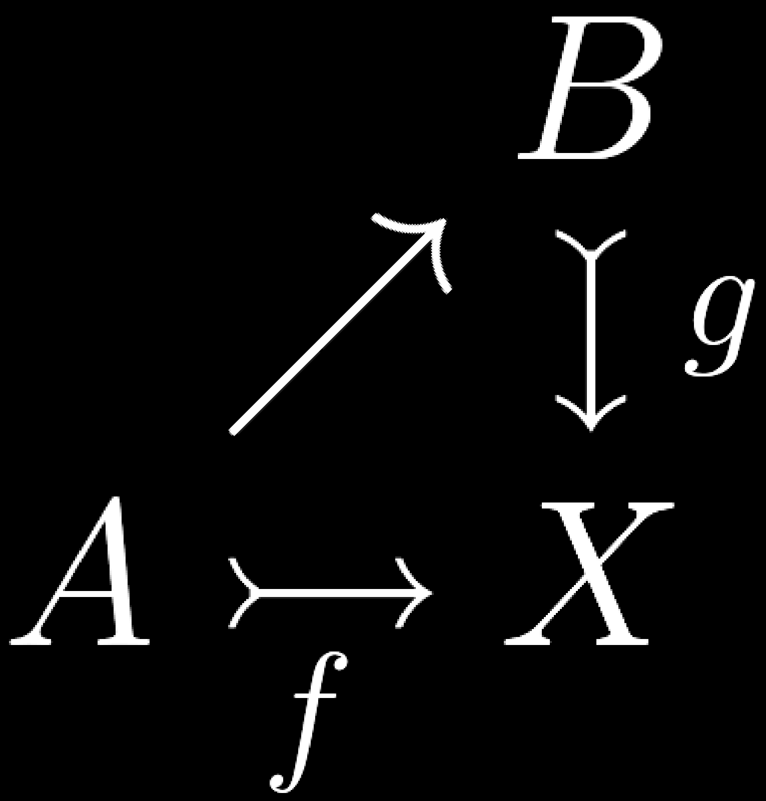
$$f : A \rightarrowtail X \qquad g : B \rightarrowtail X$$

$$f : A \rightarrowtail X \qquad g : B \rightarrowtail X$$

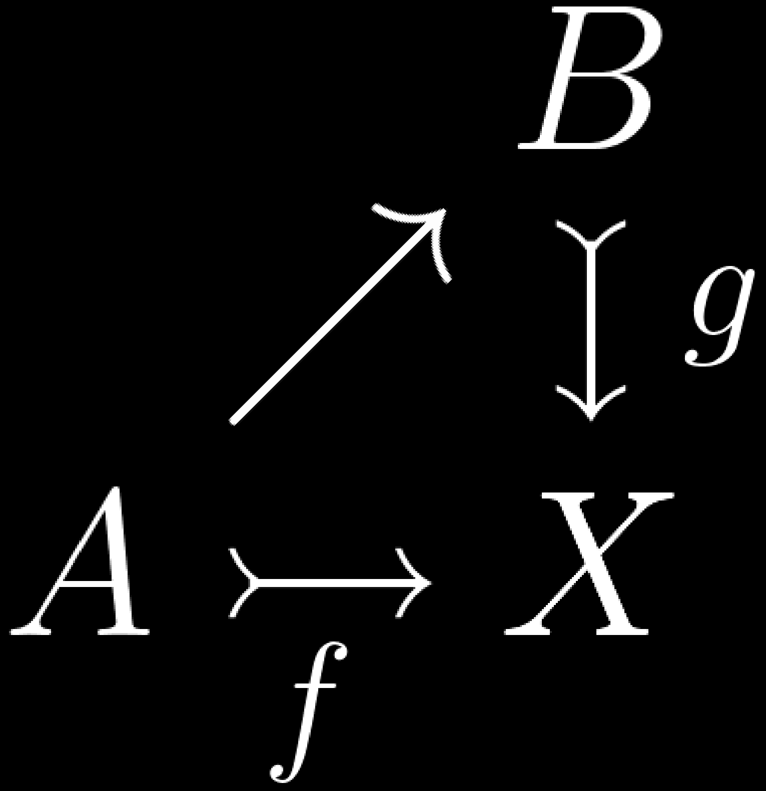
$$f \leq g$$

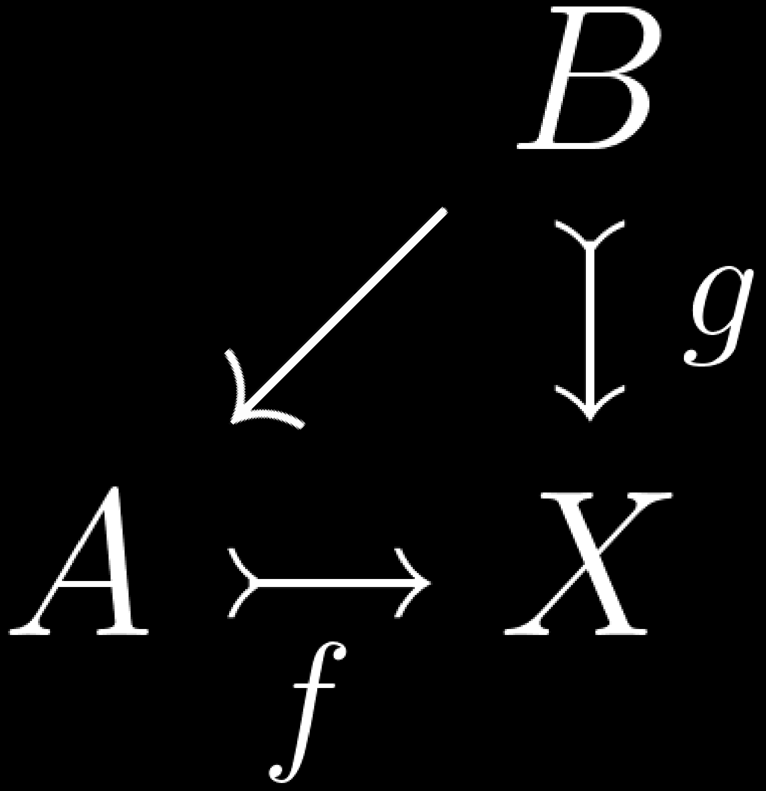
$$\begin{array}{ccc} & & B \\ & \nearrow & \big\downarrow\rlap{\scriptstyle g} \\ A & \underset{f}{\rightarrowtail} & X \end{array}$$

$$f \leq g$$

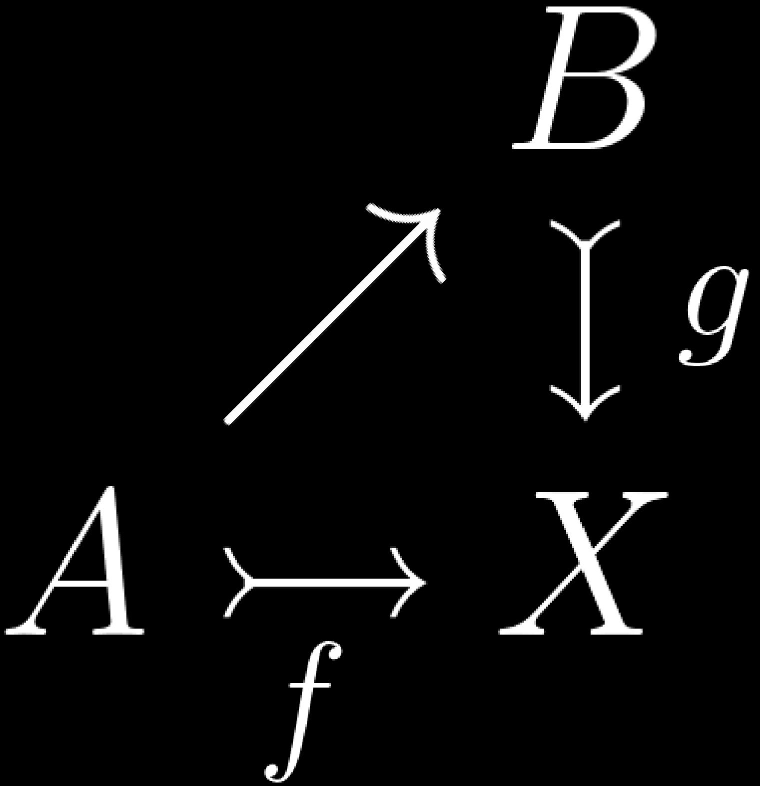


$$f \leq g \qquad\qquad g \leq f$$

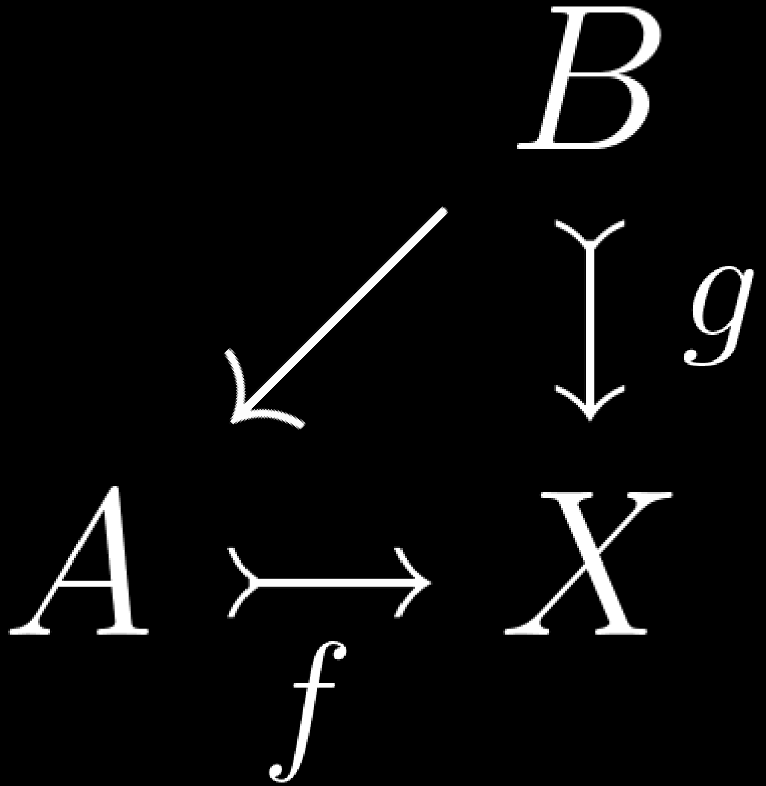




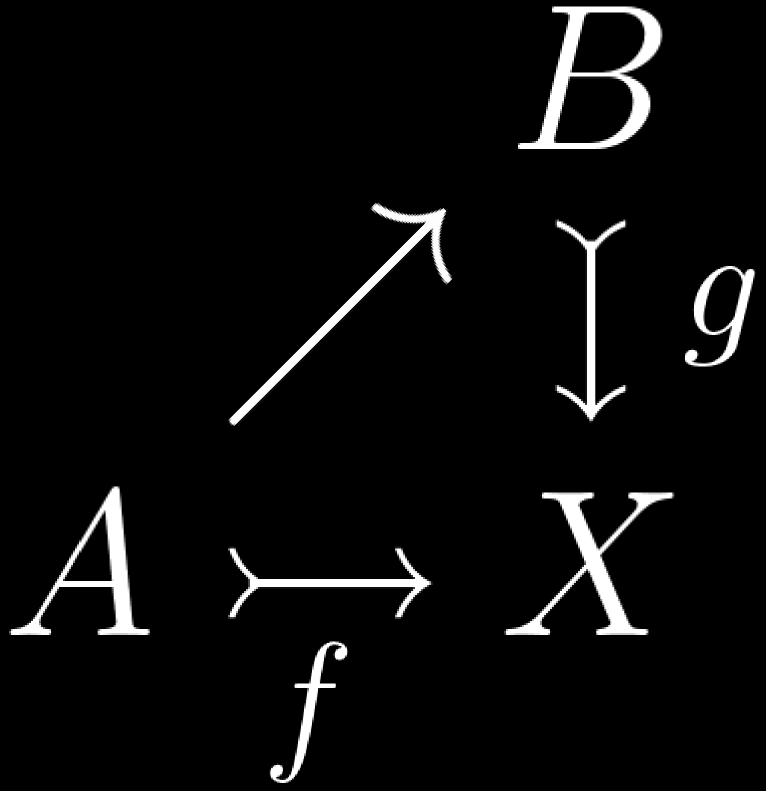
$$f \leq g$$

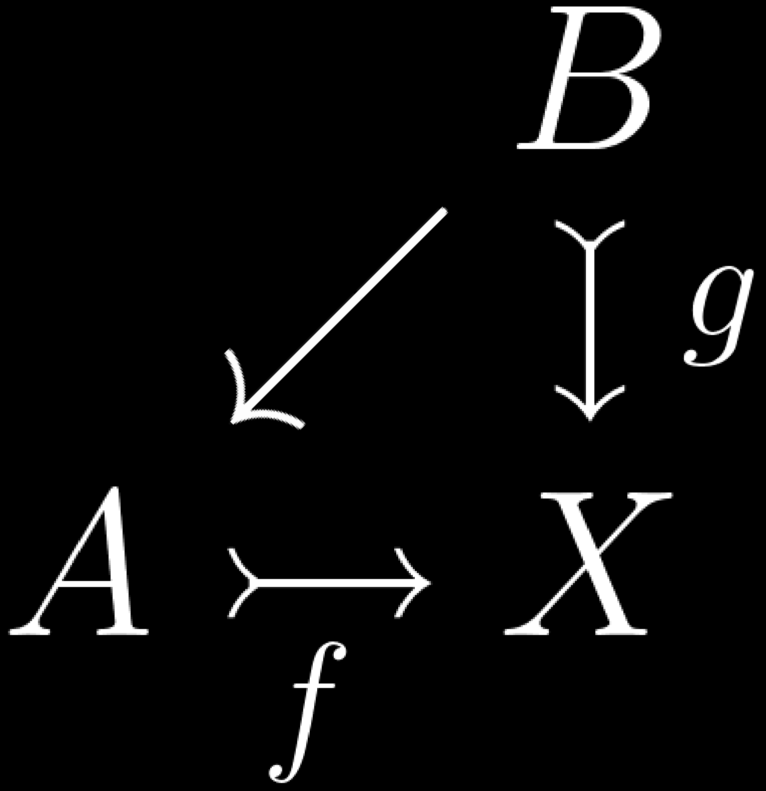


$$g \leq f$$



$$f \cong g$$





A *subobject* of an object X is an isomorphism class of monomorphisms into X.

Given a monomorphism

$$S : A \rightarrowtail X$$

we write

$$[S] \subseteq X$$

for the subobject represented by S.

$$\{1\} \cong \{2\} \cong \{\text{cat}\}$$

$$X := \{1, 2, 3\}$$

$$f : \{1\} \rightarrowtail X : 1 \mapsto 1$$

$$g : \{2\} \rightarrowtail X : 2 \mapsto 1$$

$$h : \{\text{cat}\} \rightarrowtail X : \text{cat} \mapsto 1$$

$$f : \{1\} \rightarrowtail X : 1 \mapsto 1$$

$$g : \{2\} \rightarrowtail X : 2 \mapsto 1$$

$$h : \{\text{cat}\} \rightarrowtail X : \text{cat} \mapsto 1$$

$$k : \{1\} \rightarrowtail X : 1 \mapsto 2$$
$$f : \{1\} \rightarrowtail X : 1 \mapsto 1$$

“is $\mathbb{Z} \subset \mathbb{R}$?”

Material set theory:

$\mathbb{Z}$

Equivalence classes of natural numbers

$\mathbb{R}$

Equivalence classes of Cauchy sequences

Or Dedekind cuts.

Or ultrafilters on $\mathbb{N}$.

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Equivalence classes of Cauchy sequences

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Or ultrafilters on $\mathbb{N}$.

So, $\mathbb{Z} \not\subset \mathbb{R}$.

Structuralism:

Asking if $\mathbb{Z} \subset \mathbb{R}$ or not because of their *elements* is not the right question.

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Asking if $\mathbb{Z} \subset \mathbb{R}$ or not because of their *elements* is not the right question.

Rather, ask if there is a map $\mathbb{Z} \to \mathbb{R}$ that *witnesses* that $\mathbb{Z} \subset \mathbb{R}$.

We write

$$[f] \subseteq_X [g]$$

if (any) representing monomorphisms satisfy

$$f \leq g$$

We say that an element $x \in X$ *is a member* of a subset $a \subseteq X$ and write $x \in_X a$ if x factors through a.

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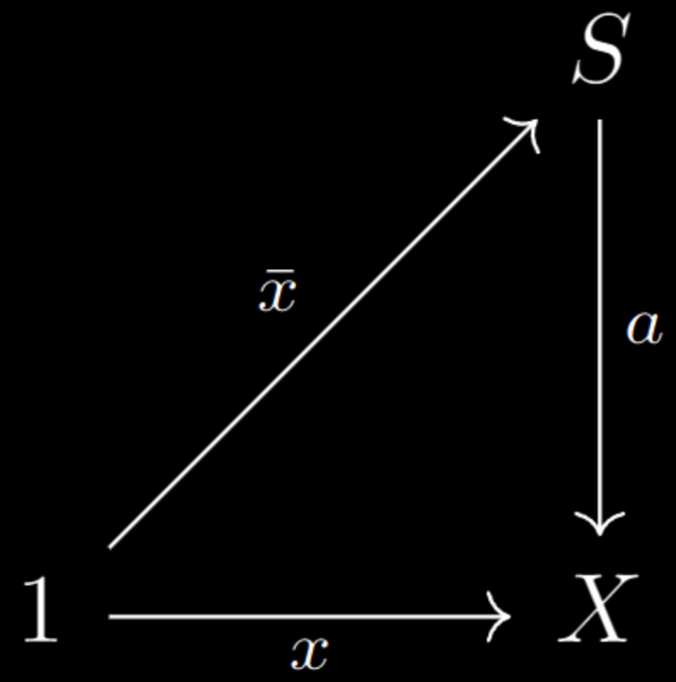


The Subobject Classifier

$$A \subseteq X$$

$$\Updownarrow$$

$$\chi_A : X \to 2$$

where $2 = \{\top, \bot\}$

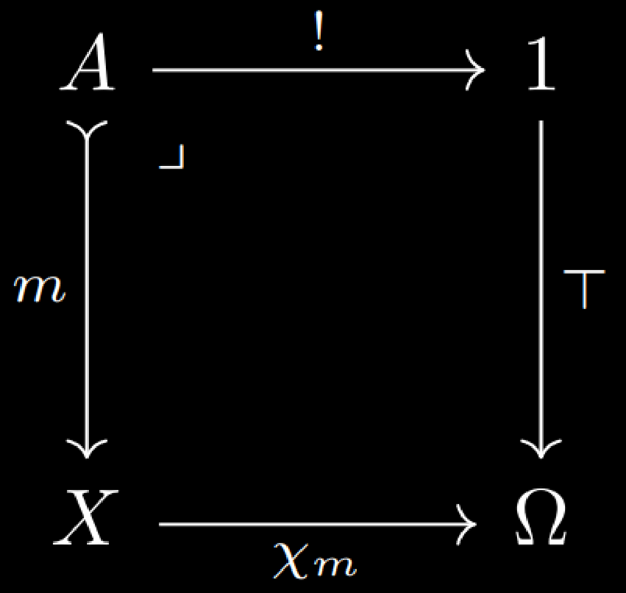
$$A \subseteq X$$

$$\chi_A(x) = \begin{cases} \top & x \in A \\ \bot & x \notin A \end{cases}$$

$$\chi_A \overset{\cong}{\mapsto} A := \chi_A^{-1}[\{\top\}]$$

$$
\begin{array}{ccc}
A = f^{-1}[\{\top\}] & \xrightarrow{\ !\ } & 1 \\
\Big\downarrow & \lrcorner & \Big\downarrow \top \\
X & \xrightarrow[\ \chi\ ]{} & 2
\end{array}
$$

A *subobject classifier* in a category $\mathscr{C}$ is an object Ω and a map $\top : 1 \to \Omega$ such that for every monomorphism $m : A \rightarrowtail X$, there exists a unique morphism $\chi_m : X \to \Omega$ such that



is a pullback square.

$$\mathrm{Sub}(X)$$

$$f : X \to Y$$

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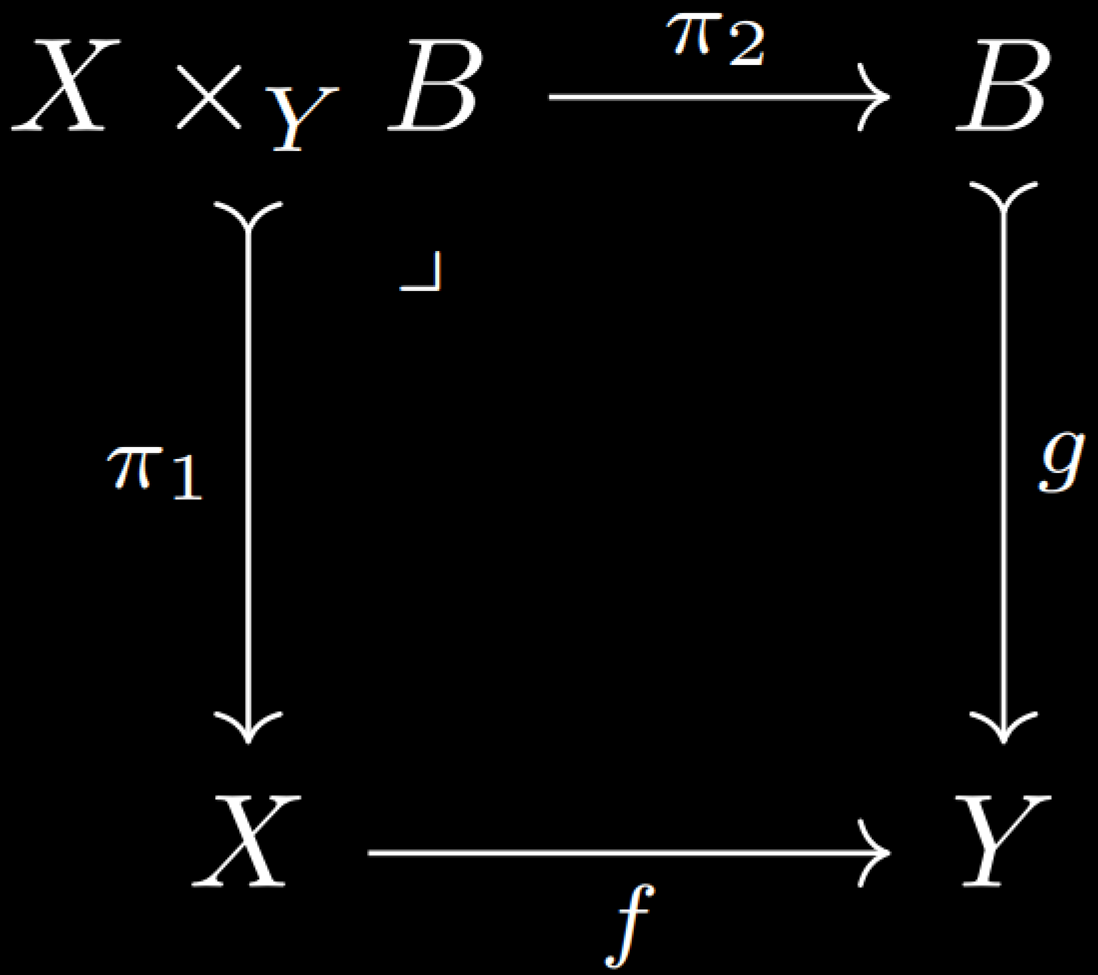
$$f : X \to Y$$

$$f^* : \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$$


$X \times_Y B$
π_2
B
π_1
g
X
f
Y

$$\mathrm{Sub} : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$$

$$\mathrm{Sub} : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$$

$$\mathrm{Sub}(X) \cong \hom(X, \Omega)$$

$$A \rightarrowtail X \qquad B \rightarrowtail X$$

$$A \cap_X B := f \times_X g$$

$$A \cup_X B := f \amalg_X g$$

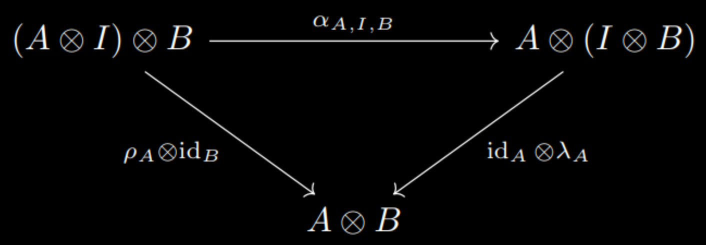
Monoidal Categories

A *monoidal category* $(\mathscr{C}, \otimes, I, \alpha, \lambda, \rho)$ consists of:

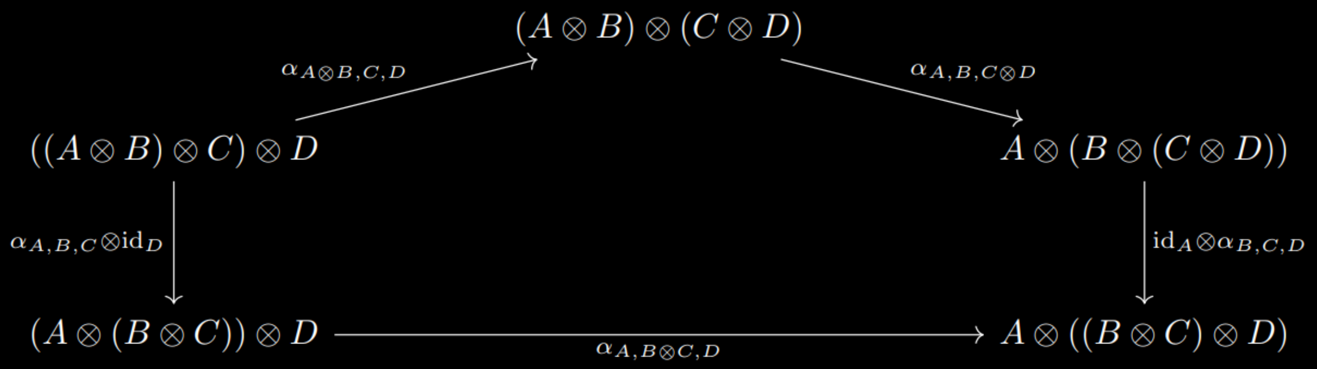
- A category $\mathscr{C}$;
- A bifunctor $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ called the *tensor product*, written in infix notation;
- A designated object I in $\mathscr{C}$ called the *unit*;
- A natural isomorphism $\alpha : ((-) \otimes (-)) \otimes (-) \Rightarrow (-) \otimes ((-) \otimes (-))$ with components of the form $\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$ called the *associator*;
- A natural isomorphism $\lambda : I \otimes (-) \Rightarrow (-)$ with components of the form $\lambda_A : (I \otimes A) \to A$ called the *left unitor*;
- A natural isomorphism $\rho : (-) \otimes I \Rightarrow (-)$ with components of the form $\rho_A : (A \otimes I) \to A$ called the *right unitor*;

subject to the *coherence conditions* that the following diagrams commute:

- the *triangle identity*:



- the *pentagon identity*:



$$(a \times e) \times b$$

$$(a \times e) \times b = a \times b$$

$$(a \times e) \times b = a \times b$$

$$(a \times e) \times b = a \times (e \times b) = a \times b$$

$$(a \times e) \times b = a \times b$$

$$(a \times e) \times b = a \times (e \times b) = a \times b$$

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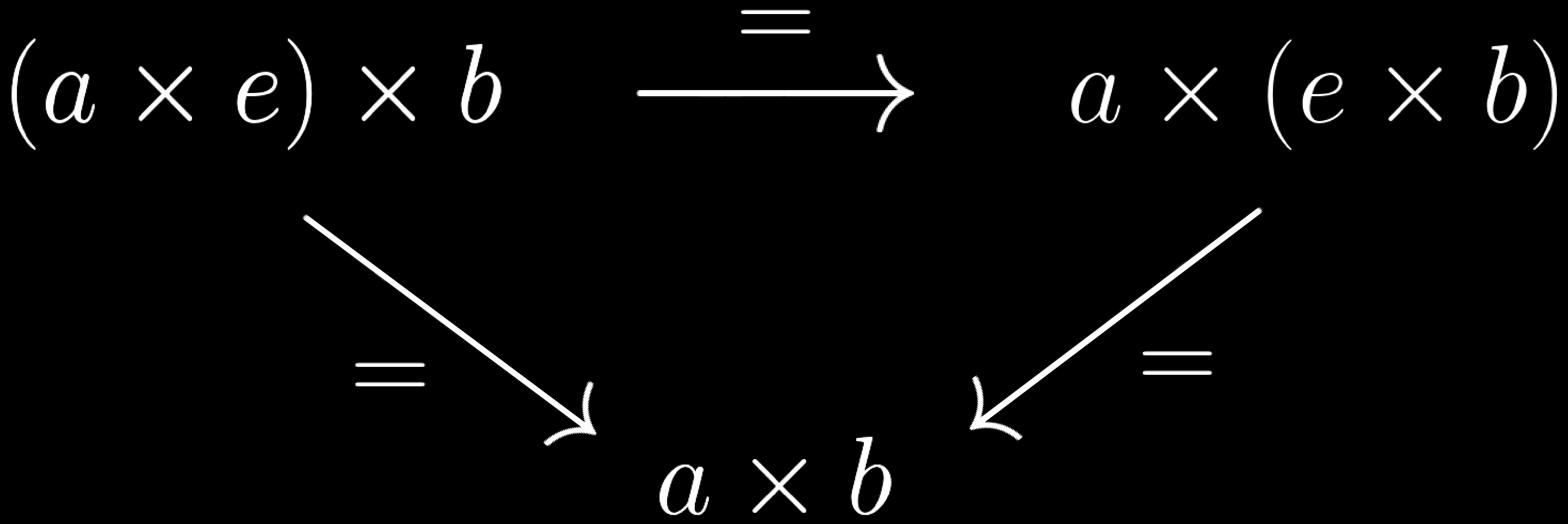
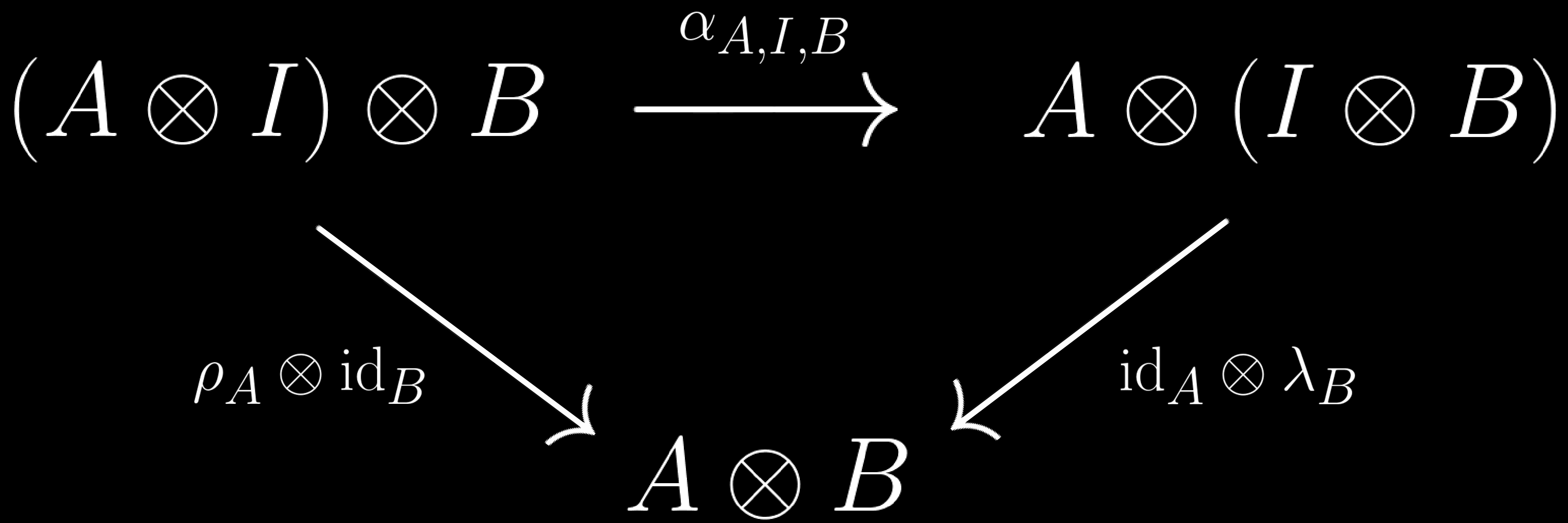
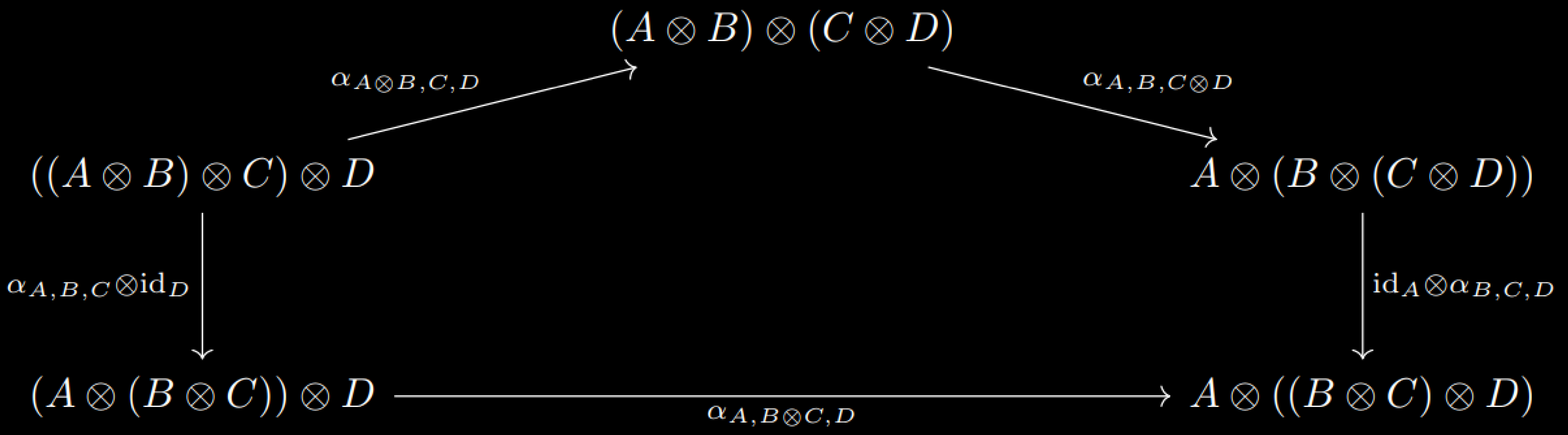

$(a \times e) \times b$
$=$
$a \times (e \times b)$
$=$
$=$
$a \times b$


$(A \otimes I) \otimes B$
$\alpha_{A,I,B}$
$A \otimes (I \otimes B)$
$\rho_A \otimes \mathrm{id}_B$
$\mathrm{id}_A \otimes \lambda_B$
$A \otimes B$


$(A \otimes B) \otimes (C \otimes D)$
$\alpha_{A \otimes B, C, D}$
$\alpha_{A, B, C \otimes D}$
$((A \otimes B) \otimes C) \otimes D$
$A \otimes (B \otimes (C \otimes D))$
$\alpha_{A,B,C} \otimes \mathrm{id}_D$
$\mathrm{id}_A \otimes \alpha_{B,C,D}$
$(A \otimes (B \otimes C)) \otimes D$
$A \otimes ((B \otimes C) \otimes D)$
$\alpha_{A, B \otimes C, D}$

$$(A \times B) \times C \cong A \times (B \times C)$$

$$1 \times A \cong A \qquad A \times 1 \cong A$$

$$(A \times B) \times C \cong A \times (B \times C)$$

$$1 \times A \cong A \qquad A \times 1 \cong A$$

$$(A \sqcup B) \sqcup C \cong A \sqcup (B \sqcup C)$$

$$\emptyset \sqcup A \cong A \qquad A \sqcup \emptyset \cong A$$

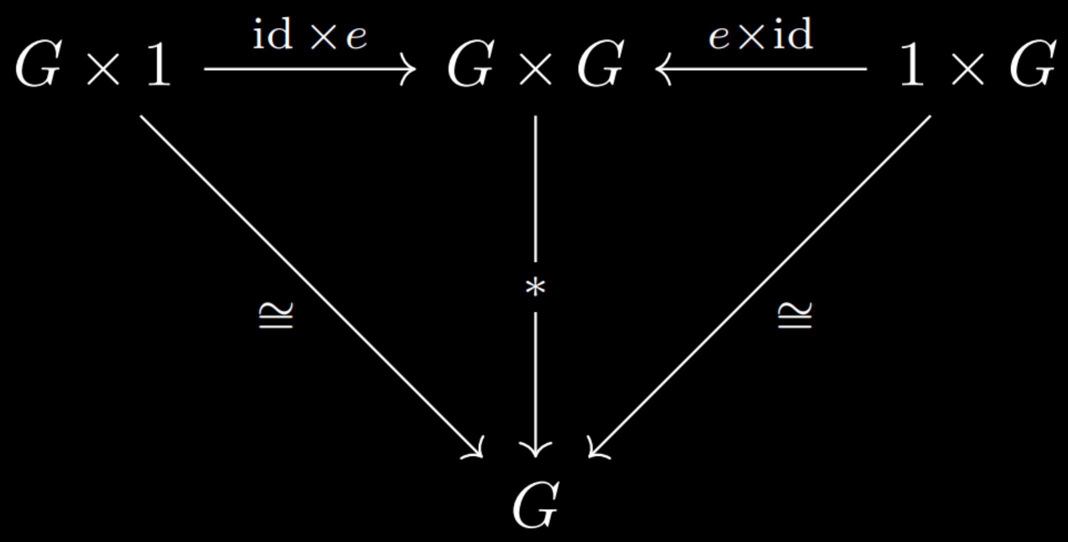
Internalisation

A *group* $(G, *)$ is a set G equipped with a binary operation $* : G \times G \to G$ that is associative, admits an identity element $e \in G$ (is *unitary*), and every element $g \in G$ has an inverse $g^{-1} \in G$ under $*$.

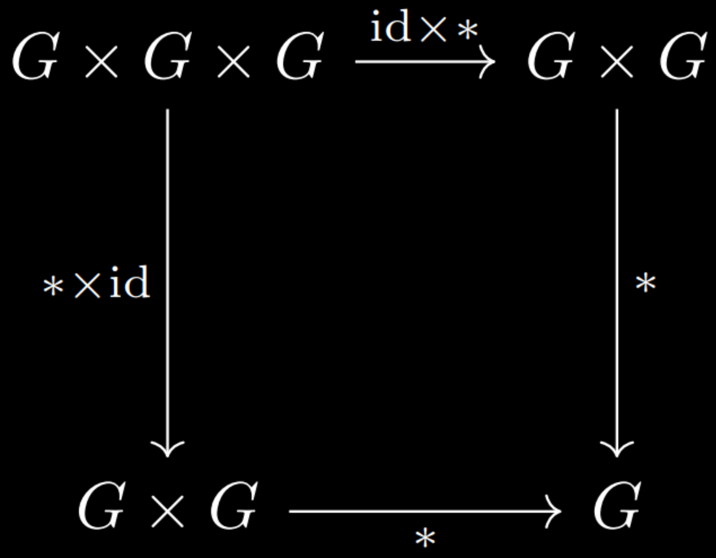
$$* : G \times G \to G$$

$$e : 1 \to G$$

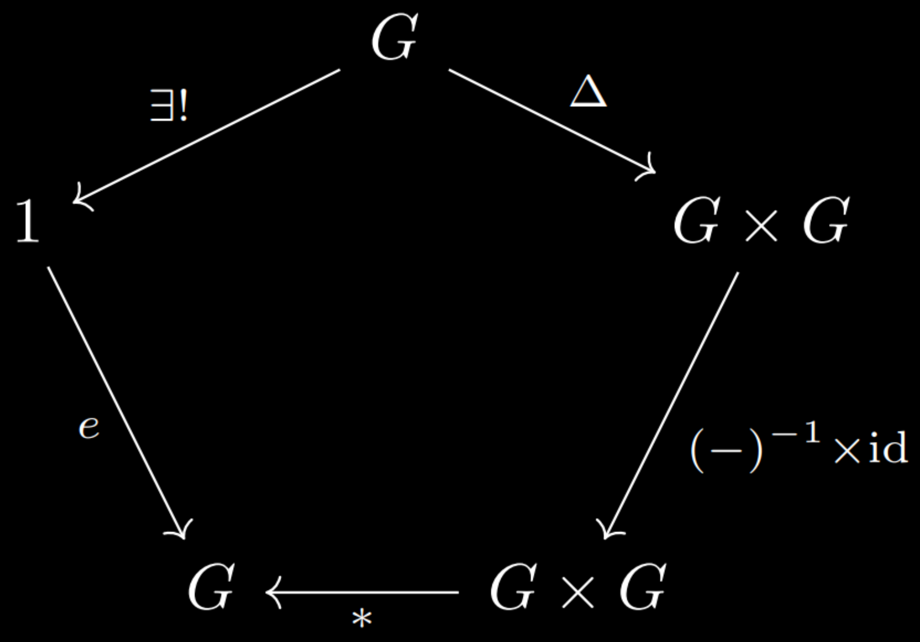
$$(-)^{-1} : G \to G$$



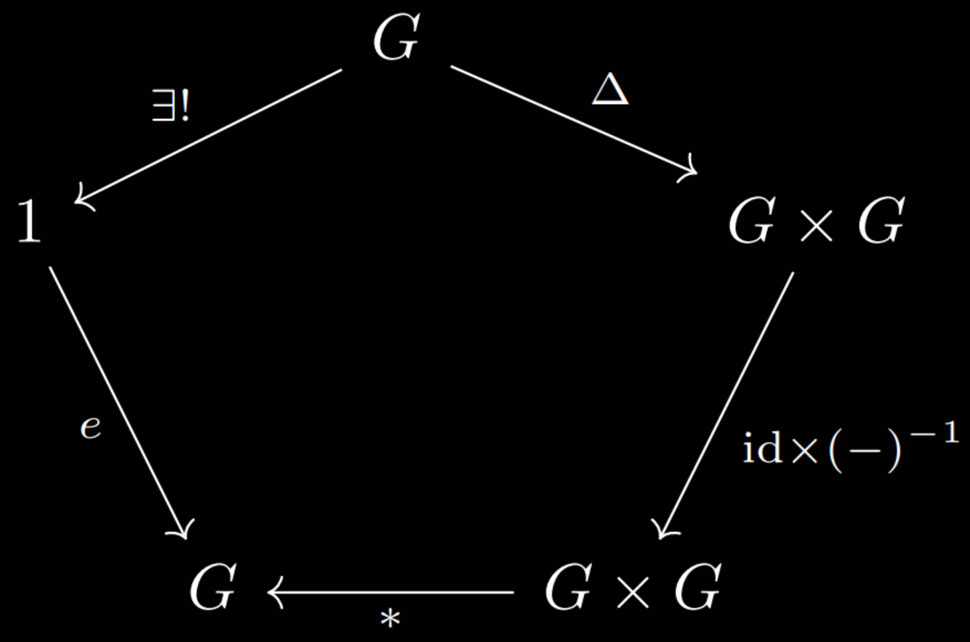
G × 1
id × e
G × G
e × id
1 × G
≅
*
≅
G



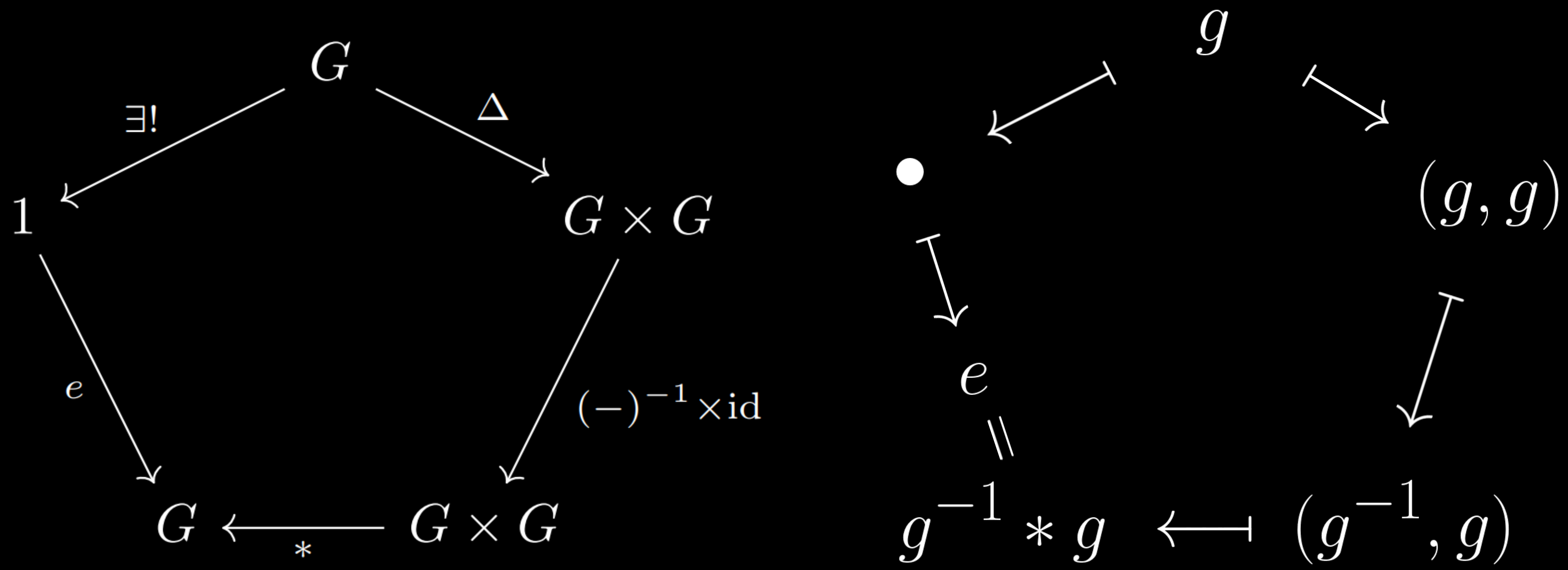
G × G × G
id × *
G × G
* × id
*
G × G
*
G

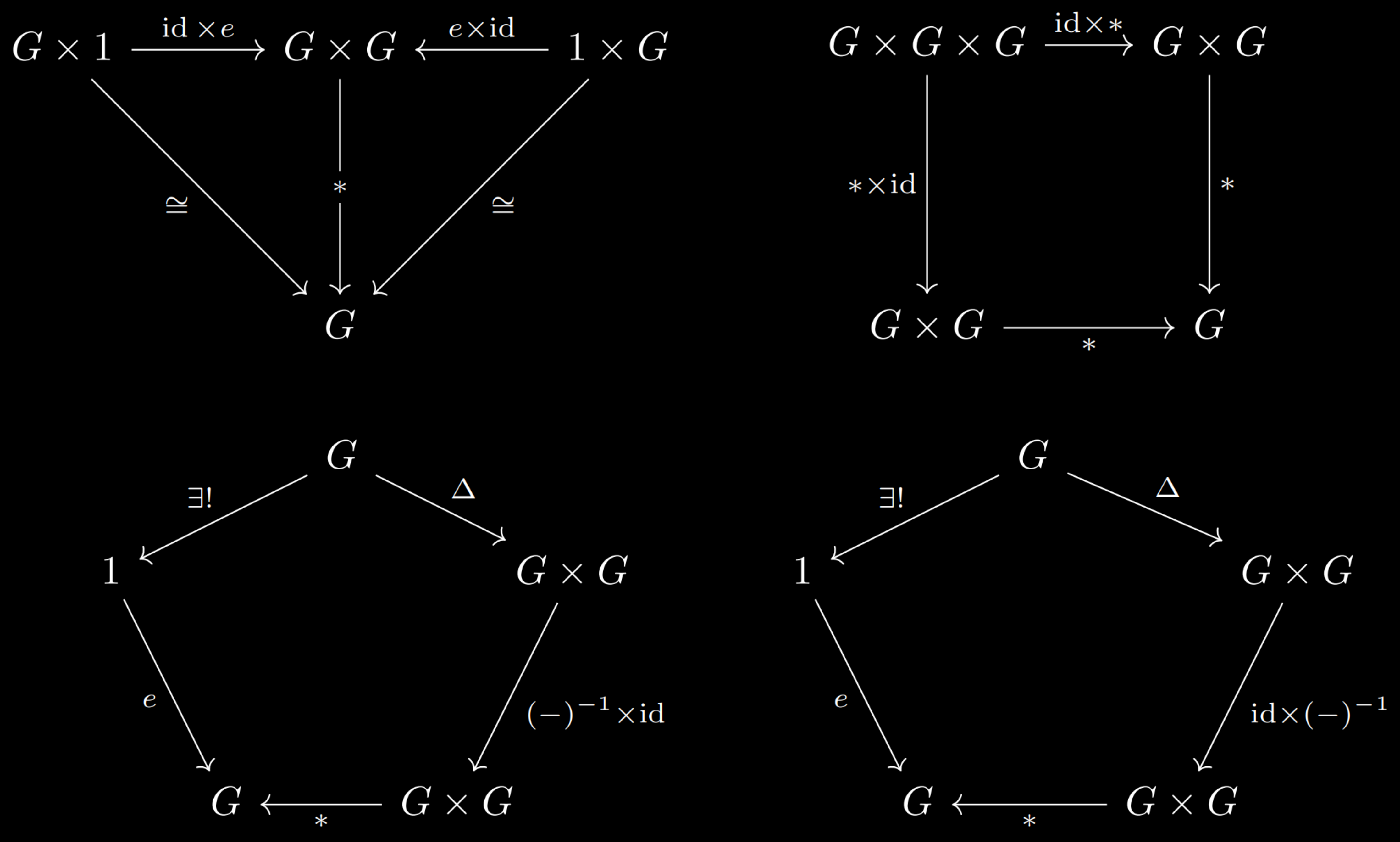


G
∃!
Δ
1
G × G
e
(−)^{−1} × id
G
*
G × G



G
∃!
Δ
1
G × G
e
id × (−)^{−1}
G
*
G × G


G
∃!
Δ
1
G × G
e
(−)^{−1} × id
G
*
G × G
g
•
(g, g)
e
g^{−1} * g
(g^{−1}, g)


G × 1
id × e
G × G
e × id
1 × G
≅
*
≅
G
G × G × G
id × *
G × G
* × id
*
G × G
*
G
G
∃!
Δ
1
G × G
e
(−)^{−1} × id
G
*
G × G
G
∃!
Δ
1
G × G
e
id × (−)^{−1}
G
*
G × G

An *internal group* in a category $\mathscr{C}$ that admits finite products is an object G equipped with morphisms

$$* : G \times G \to G$$

$$e : 1 \to G$$

$$(-)^{-1} : G \to G$$

such that the previous diagrams all commute.

$\mathscr{C}$	internal groups
$\mathbf{Set}$	ordinary groups
$\mathbf{Top}$	topological groups
$\mathbf{Man}^{\infty}$	Lie groups
$\mathbf{Grp}$	abelian groups
$\vdots$	$\vdots$

$(\mathscr{C}, \otimes, I)$	internal monoids
$(\mathbf{Mon}, \times, 1)$	commutative monoid
$(\mathbf{Ab}, \otimes_{\mathbb{Z}}, \mathbb{Z})$	(unital) ring
$(R\text{-}\mathbf{Mod}, \otimes_R, R)$	R-algebra
$(\mathbf{Cat}, \times, \mathbb{1})$	strict monoidal category
$\vdots$	$\vdots$

Internal Homs

A

A, B

$$[A, B]$$

$$X \to [A, B]$$

$$X \to [A, B]$$

$$\cong$$

$$X \times A \to B$$

$$X \to [A, B]$$

$$\cong$$

$$X \times A \to B$$

Let $\mathscr{C}$ be a monoidal category, and let A and B be objects of $\mathscr{C}$.

The *internal hom-object*, or just *internal hom*, of A and B is an object $[A, B]$ such that

$$\hom\big(X, [A, B]\big) \cong \hom(X \times A, B)$$

naturally in X.

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naturally in X.

A monoidal category is *closed monoidal* if for every object A, the right tensor by A has a right adjoint:

$$(-) \otimes A \dashv [A, -]$$

so

$$\hom\big(X, [A, B]\big) \cong \hom(X \otimes A, B)$$

naturally in all 3 variables.

A closed monoidal category that is cartesian monoidal is called *cartesian closed*.

$$\hom\left(X, [A, B]\right) \cong \hom(X \times A, B)$$

A closed monoidal category that is cartesian monoidal is called *cartesian closed*.

Example. Any locally small category has a set of morphisms between any two objects.

Set is locally small. So, **Set** is cartesian closed.

In a cartesian closed category, we write

$$B^A$$

for the internal hom-object

$$[A, B]$$

and call it an *exponential object.*

This notation is compatible with the categorical product in that

$$A^1 \cong A$$

$$A^2 \cong A \times A$$

$$A^3 \cong A \times A \times A$$

$$\vdots$$

$$A^n \cong \prod_n A$$

$$2 := 1 \amalg 1$$

$$3 := 1 \amalg 1 \amalg 1$$

$$\vdots$$

$$\hom(X, B^A) \cong \hom(X \times A, B)$$

$$\hom(X, B^A) \cong \hom(X \times A, B)$$

$$f : X \times A \to B$$

$$\hom(X, B^A) \cong \hom(X \times A, B)$$

$$f^\flat : X \to B^A \quad \longleftarrow\!\!\mid \quad f : X \times A \to B$$

$$\hom(X, B^A) \cong \hom(X \times A, B)$$

$$f^\flat : X \to B^A \quad \longleftarrow\!\!\mid \quad f : X \times A \to B$$

$$g : X \to B^A$$

$$\hom(X, B^A) \cong \hom(X \times A, B)$$

$$f^\flat : X \to B^A \quad \longleftarrow\!\!\!| \quad f : X \times A \to B$$

$$g : X \to B^A \quad \longmapsto \quad g^\sharp : X \times A \to B$$

$$\hom(X, B^A) \cong \hom(X \times A, B)$$

$$\hom(1, B^A) \cong \hom(1 \times A, B)$$

$$\hom(1, B^A) \cong \hom(1 \times A, B) \cong \hom(A, B)$$

$$\hom(1, B^A) \cong \hom(1 \times A, B) \cong \hom(A, B)$$

$$\hom(1, B^A) \cong \hom(A, B)$$

Topoi

A (*elementary*) *topos* is a category that:

- is finitely complete;
- is cartesian closed;
- has a subobject classifier.

Lemma. *Every monomorphism in a topos is regular.*

Corollary. *Every topos is balanced.*

Theorem. *Every topos is finitely cocomplete.*

Theorem. *Every morphism factors essentially uniquely through its image into the composition of an epimorphism and a monomorphism.*

A (*elementary*) *topos* is a category that:

- is finitely complete;
- is cartesian closed;
- has a subobject classifier.

Set

$\mathbf{Set}$ is non-trivial. That is, $\mathbf{Set} \not\simeq \mathbb{1}$.

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(i) $0 \ncong 1$.

If $f, g : X \to Y$ are parallel morphisms such that every morphism $x : 1 \to X$ equalises f and g, then $f = g$.

If $f, g : X \to Y$ are parallel morphisms such that every morphism $x : 1 \to X$ equalises f and g, then $f = g$.

(ii) The terminal object 1 is a separator.

(i) $0 \ncong 1$.

(ii) The terminal object 1 is a separator.

A topos that satisfies (i) and (ii) is called *well-pointed*.

X

$$x \in X$$

$x \in X \qquad r : X \to X$

$$x \in X \qquad r : X \to X$$

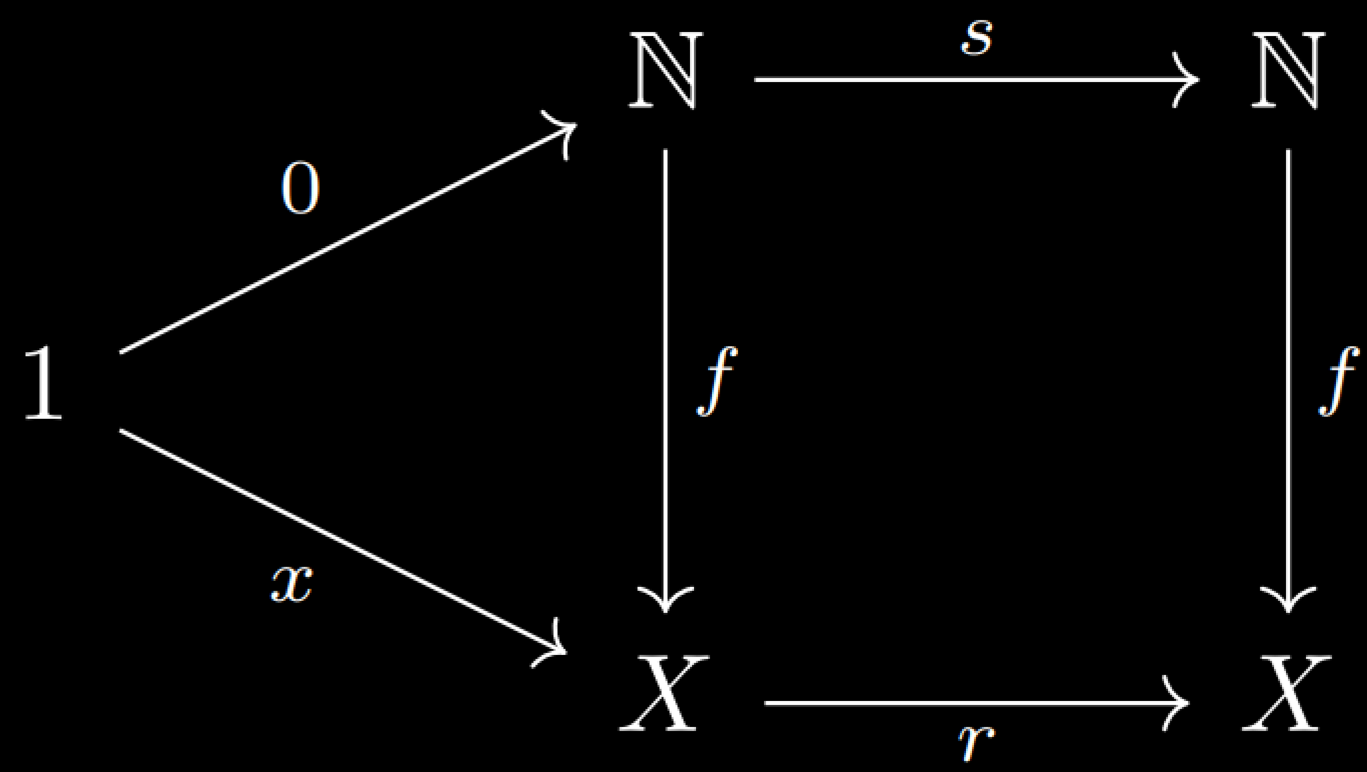
$$(x_i)_{i=0}^{\infty} \subseteq X$$

$$x \in X \qquad r : X \to X$$

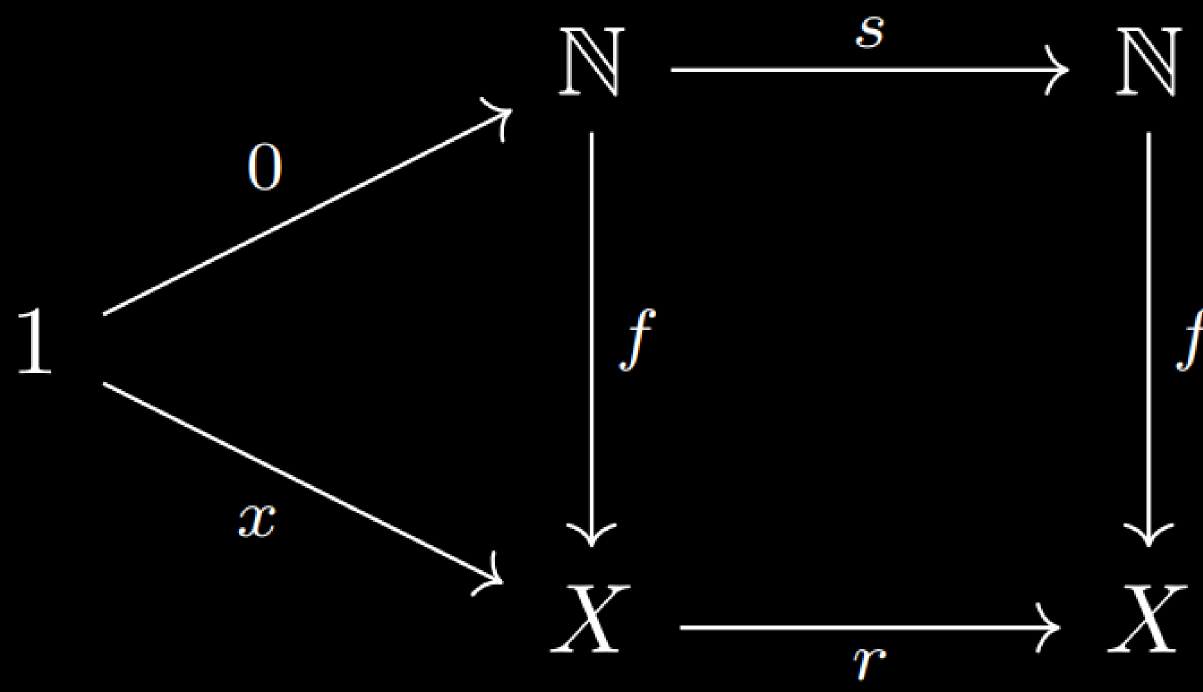
$$(x_i)_{i=0}^{\infty} \subseteq X \qquad \begin{gathered} x_0 = x \\ x_{n+1} = r(x_n) \end{gathered}$$

$$x \in X \qquad r : X \to X$$

$$(x_i)_{i=0}^{\infty} = f : \mathbb{N} \to X$$


1
0
x
ℕ
s
ℕ
f
f
X
r
X

A *natural numbers object* is a triple $(\mathbb{N}, 0, s)$ consisting of an object $\mathbb{N}$, an element $0 : 1 \to \mathbb{N}$, and a *successor morphism* $s : \mathbb{N} \to \mathbb{N}$ with the universal property that the triple $(\mathbb{N}, 0, s)$ factors through every other triple $(X,\ x,\ r)$ uniquely:



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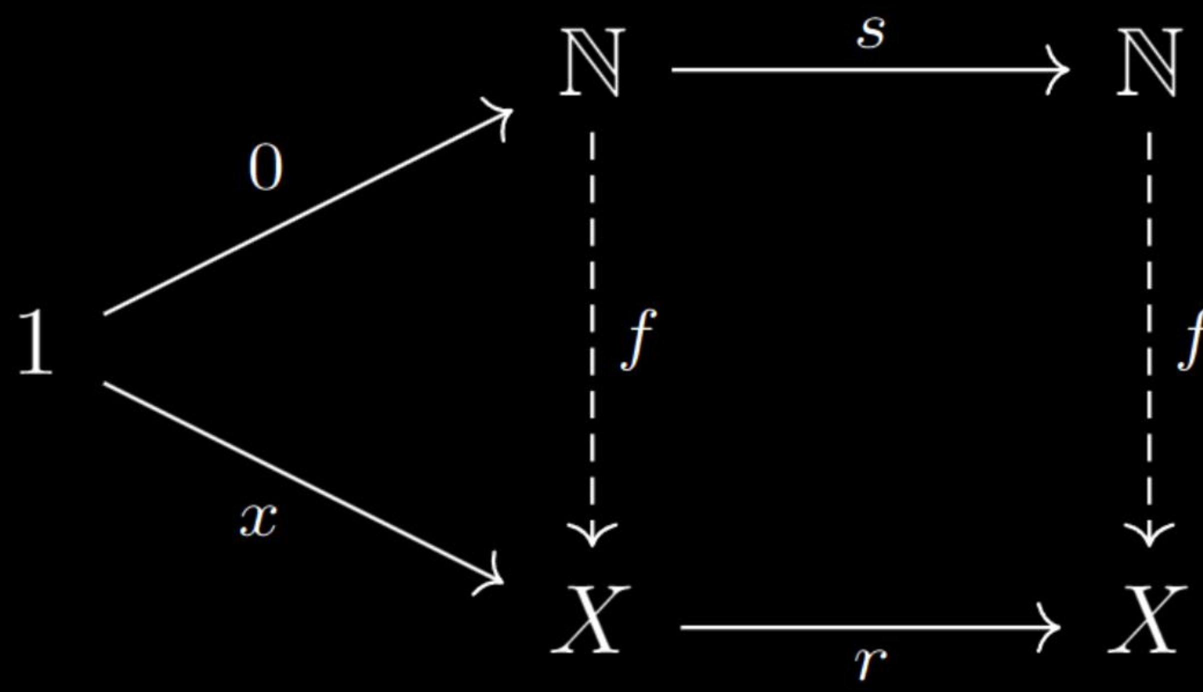

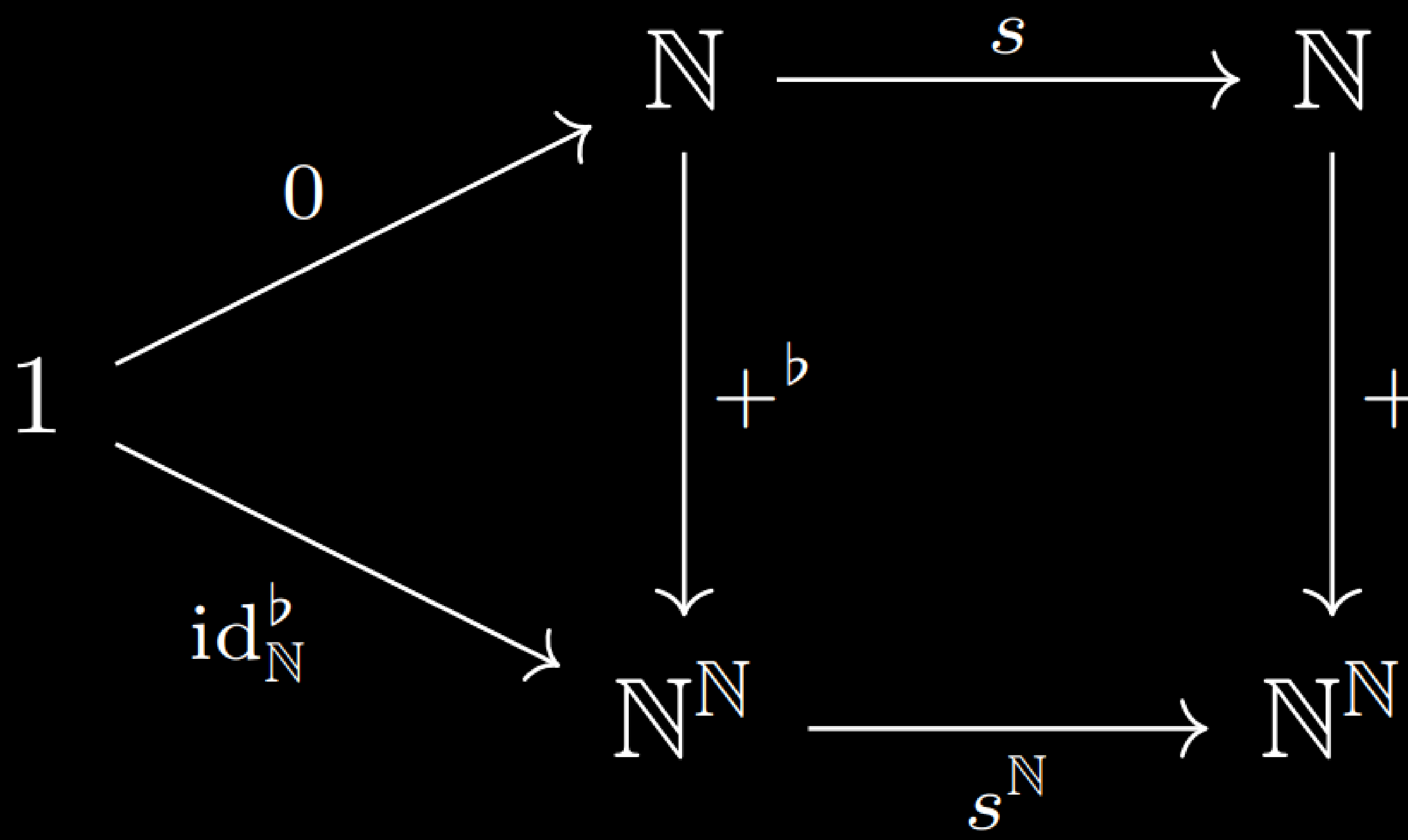

1
0
$\mathbb{N}$
s
$\mathbb{N}$
$+^\flat$
$+^\flat$
$\mathrm{id}_{\mathbb{N}}^\flat$
$\mathbb{N}^{\mathbb{N}}$
$s^{\mathbb{N}}$
$\mathbb{N}^{\mathbb{N}}$

(*iii*) **Set** has a natural numbers object.

$$f : A \twoheadrightarrow B$$

$$f : A \twoheadrightarrow B \qquad s : B \rightarrowtail A$$

$$f \circ s = \mathrm{id}_A$$

$$f : A \twoheadrightarrow B \qquad s : B \rightarrowtail A$$

$$f \circ s = \mathrm{id}_A$$

(iv) Epimorphisms split.

(i) $0 \not\cong 1$.

(ii) The terminal object 1 is a separator.

(iii) There is a natural numbers object.

(iv) Epimorphisms split.

Sets and set functions form a well-pointed topos with natural numbers object and Choice.

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1. Function composition is associative and has identities
2. There exists an empty set
3. There exists a singleton set
4. Functions are completely characterised by their action on elements
5. Given sets X and Y, we may form their cartesian product $X \times Y$
6. Given sets X and Y, we may form the set of functions from X to Y
7. Given a function $f : X \to Y$ and $y \in Y$ we may form the fibre $f^{-1}[y]$
8. The subsets of a set X correspond to the functions $X \to \{0, 1\}$
9. The natural numbers form a set
10. Every surjection admits a section

Material and Structural Sets

Material-sets	Structural-sets

Material-sets

determined by elements up to equality

Structural-sets

determined by generalised elements up to isomorphism, but subsets up to equality

Material-sets

determined by elements up to equality

elements are always sets

Structural-sets

determined by generalised elements up to isomorphism, but subsets up to equality

elements are never sets

Material-sets	Structural-sets
determined by elements up to equality	determined by generalised elements up to isomorphism, but subsets up to equality
elements are always sets	elements are never sets

Material-sets	Structural-sets
determined by elements up to equality	determined by generalised elements up to isomorphism, but subsets up to equality
elements are always sets	elements are never sets
lots of side effects from constructions	abstract structures encapsulate and isolate properties without side effects

$$A \in X$$

“for all $x \in \mathbb{R}, x^2 \geq 0$”

$$\text{“for all } x \in \mathbb{R}, x^2 \geq 0\text{”}$$

“for all things x, if x is a real number, then $x^2 \geq 0$”

"for all $x \in \mathbb{R}, x^2 \geq 0$"

"for all things x, if x is a real number, then $x^2 \geq 0$"

$$\forall x : x \in \mathbb{R} \to x^2 \geq 0$$

“for all $x \in \mathbb{R}, x^2 \geq 0$”

“for all things x, if x is a real number, then $x^2 \geq 0$”

$$\forall x : x \in \mathbb{R} \to x^2 \geq 0$$

“it is a property of all real numbers x that $x^2 \geq 0$”

“for all $x \in \mathbb{R}, x^2 \geq 0$”

“for all things x, if x is a real number, then $x^2 \geq 0$”

$$\forall x : x \in \mathbb{R} \to x^2 \geq 0$$

“it is a property of all real numbers x that $x^2 \geq 0$”

$$\forall x \in \mathbb{R} : x^2 \geq 0$$

$$\text{“for all } x \in \mathbb{R}, x^2 \geq 0\text{”}$$

$$\text{“for all things } x\text{, if } x \text{ is a real number, then } x^2 \geq 0\text{”}$$

$$\forall x : x \in \mathbb{R} \to x^2 \geq 0$$

$$\rightsquigarrow$$

$$\text{“if } \mathbb{Q}[x] \text{ is a real number, then } \left(\mathbb{Q}[x]\right)^2 \geq 0\text{”}$$

$$L = \left\{ z \in \mathbb{C} : \Re(z) = \frac{1}{2} \right\}$$

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"for all $z \in \mathbb{C}$, if $\zeta(z) = 0$ and z is not a negative even integer, then $z \in L$"

$$L = \left\{ z \in \mathbb{C} : \Re(z) = \frac{1}{2} \right\}$$

i.e. functions
$1 \to \mathbb{C}$

"for all $z \in \mathbb{C}$, if $\zeta(z) = 0$ and z is not a negative even integer, then $z \in L$"

does z factor through L?

Material-sets	Structural-sets
determined by elements up to equality	determined by generalised elements up to isomorphism, but subsets up to equality
elements are always sets	elements are never sets
lots of side effects from constructions	abstract structures encapsulate and isolate properties without side effects
propositional membership only	type-declaration membership; supports propositional patterns in the presence of ambient sets.